

# DONALDSON-THOMAS THEORY

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THESIS FOR THE DEGREE OF  
**MASTER IN MATHEMATICS**

(MASTER OF SCIENCE)



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MAY 2010



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# Introduction

The general problem of counting the number of geometric objects of a given type is classical in algebraic geometry, and these questions make up the field of enumerative geometry. In the early 90's the question of counting curves in a given ambient space received much attention due to the discovery that ideas from string theory could be applied successfully to these problems. One result of this development was the introduction of Gromov-Witten invariants. These are numbers associated to a scheme  $X$  which, roughly speaking, count the number of maps of curves to  $X$  with given genus and homology class in  $X$ .

In especially nice cases, these invariants do coincide with what one would expect from a classical count of curves, supposing this is defined, and Gromov-Witten theory has led to new answers to questions from classical enumerative geometry. One famous example is Kontsevich's complete solution of the problem of counting the number of rational nodal curves of degree  $d$  passing through  $3d - 2$  general points in the plane.

In general, however, the Gromov-Witten invariants behave rather differently from a naive count, and their enumerative meaning, if any, is not always obvious. For example, the Gromov-Witten count of curves might be a finite number when the naive count would be infinite, and the invariants can take both rational and negative numbers.

On the other hand Gromov-Witten invariants are in important ways better behaved than the ordinary counts. One example is the fact that they are invariant under deformations of the space  $X$  on which one is counting, something which fails for the naive count.

The case of counting curves lying on a threefold turns out to be of particular interest. In 2000, Thomas proved [28] the existence of a new invariant counting curves on a threefold, later known as Donaldson-Thomas invariants. These share many properties with the Gromov-Witten invariants, and are constructed in a similar way. The important difference between the two curve counts lies in what one means by the word curve. In Donaldson-Thomas theory one takes curves to mean one-dimensional subschemes of  $X$ , in contrast to the Gromov-Witten meaning of maps of curves to  $X$ .

It the later (2006) articles [21] and [22] by Maulik, Nekrasov, Okounkov and Pandharipande three conjectures were posed about Donaldson-Thomas

invariants. One of them stated that the Gromov-Witten invariants and Donaldson-Thomas invariants of a threefold  $X$ , are essentially equivalent. These conjectures are referred to as the MNOP conjectures, and the conjectured equivalence between the two curve counts is known as the GW/DT-correspondence.

Yet other curve counting invariants exist, both physical and mathematical, but we will restrict our attention to the two we have mentioned so far. Of the two, our focus will be on Donaldson-Thomas invariants, restricting mostly to definitions and examples in Gromov-Witten theory.

The text is divided into three chapters. In the first chapter we present the general theory used in the construction of both invariants. We try to motivate the definition of the invariants, as well as give the precise definitions of both invariants. We also give the formulation of the MNOP conjectures.

In the second chapter we present some techniques for calculating Donaldson-Thomas invariants. The first is known as toric localization, and is applicable in the cases where there is a group action of a torus  $T \cong (\mathbb{C}^*)^n$  on  $X$ . We present part of an article showing how this technique can be used to calculate Donaldson-Thomas invariants of a toric threefold.

We also give a presentation of the local Donaldson-Thomas theory of curves, where the threefold on which we count is a rank two bundle over a smooth proper curve  $C$ . Ordinarily this invariant would not be well defined, as the moduli space of curves on this threefold is nonproper, but we shall see how invariants can still be defined via localization.

The second important technique goes by expressing Donaldson-Thomas invariants as a certain weighted sum of Euler characteristics. This can in some cases allow stratification arguments to be used in calculating Donaldson-Thomas invariants. This tool is however only applicable in the case where the threefold  $X$  has trivial canonical class. We present an article demonstrating this technique in action, obtaining expressions for some of the Donaldson-Thomas invariants of a quintic threefold.

The final chapter concerns the problem of computing Donaldson-Thomas invariants of a threefold  $X$  that admits a locally trivial elliptic fibration, i.e. that admits a morphism to a surface  $S$  such that all fibers are isomorphic to a fixed elliptic curve  $E$ . In the article [10] Edidin and Qin calculated some of the Donaldson-Thomas invariants of a product threefold  $E \times S$ . Using the expression of Donaldson-Thomas invariants as a weighted Euler characteristic we extend some of these results to Calabi-Yau threefolds admitting locally trivial elliptic fibrations.

## Conventions and cautions

All schemes are over  $\mathbb{C}$ . Throughout the text we shall reserve the letter  $X$  to denote the smooth, projective threefold on which we count curves.

By saying that a scheme Calabi-Yau we mean that it has trivial canonical class.

In the first chapter, we generally ignore the possibility that our moduli schemes could instead be stacks, even though most of the constructions and results work just as well in this more general setting. In later chapters we are working with Donaldson-Thomas theory, where the moduli space is the Hilbert scheme, so stackiness is not an issue.

I would like to express my thanks towards my advisor, Prof. Ragni Piene, for suggesting the topic of this thesis, and otherwise providing valuable guidance through the entire writing process. Gentle reminders of the approaching deadline months before it would have crossed my mind have probably been essential.





# Chapter 1

## Background material

We begin with describing the common framework of the two curve counting theories. One starts with a moduli space  $M$  parametrizing curves on  $X$ . The basic difference between the two theories lies in the choice of the precise mathematical meaning of the expression “curve on  $X$ ”. In Donaldson-Thomas theory a curve is a closed subscheme of  $X$  of dimension not greater than 1. Thus the moduli space  $M$  is in this case a certain Hilbert scheme of  $X$ . In Gromov-Witten theory one takes curve to mean a so-called stable map from a curve  $C$  to  $X$ . This notion of a curve gives a different moduli space  $\overline{M}_{g,r}(X, \beta)$ .

Assume we are given a curve  $C$  in  $X$  represented by a point  $p$  in the moduli space  $M$ . One can use information about the space of deformations of  $C$  to find an expected dimension of  $M$  in  $p$ . For the two moduli spaces we consider this dimension will be the same at every point, and so we can assign to the entire space an expected or *virtual* dimension. This virtual dimension will in our cases depend rather simply on the discrete invariants of the curve and of the threefold  $X$ . For various reasons it will often not be the same as the usual dimension of  $M$ . It will, however, always be a lower bound for the actual dimension.

### 1.1 The virtual fundamental class

An essential part of the definition of both Gromov-Witten and Donaldson-Thomas theory is the construction of a *virtual fundamental class*, which is a homology class on the moduli space having dimension the same as the virtual dimension of the moduli space. In order to explain why we need such a class, we first describe how we could naively try to count curves, and look at what goes wrong with the simple approach.

Say we have a smooth, projective threefold  $X$ , and a moduli space of curves on it,  $M$ . If there are only finitely many curves in  $M$ , so  $M$  is 0-dimensional, we can define the count of the number of curves in  $M$  by

taking the degree of its fundamental class, e.g.  $\#(M) = \int_{[M]} 1$ . If  $M$  has positive dimension, we have to add restrictions on which curves we want to count. These restrictions are given as cohomology classes  $\gamma_i \in H^*(X)$ , typically representing subschemes of  $X$ .

These classes can then in some way, depending on which moduli space is used, be pulled back to cohomology classes on  $M$ , say  $\omega_i \in H^*(M)$ . Intuitively intersecting with  $\omega_i$  should be the same as imposing the constraint that a curve meets the subscheme corresponding to  $\gamma_i$ . If the codimensions of  $\omega_i$  add up to the dimension of  $M$  we can calculate a number,

$$\int_{[M]} \prod \omega_i,$$

which is then a count of all curves in  $M$  meeting the subschemes represented by the  $\gamma_i$ .

There are several problems with this way of doing things. First of all, we would like the numbers we get to be invariant under deformations of  $X$ , something which is not achieved by this definition. To take one example, it is well known that on a generic quintic threefold in  $\mathbb{P}^4$  there are 2875 rational curves of degree one. However, we may deform this threefold to the Fermat quintic, the threefold in  $\mathbb{P}^4$  defined by

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0,$$

The lines on this threefold are all contained in one of 50 one-dimensional families [1], in particular a naive count of lines would give an infinite number.

Another reason, related to the first, has to do with dimension. From looking at deformations and obstructions to deformations of the curves we get an expected (virtual) dimension of the moduli space, which is invariant under deformations of  $X$ . We would like to count curves as if this dimension was the actual one, but as these moduli spaces are quite irregular,  $M$  often has components of larger dimension than what we expect.

In some cases this is because compactifying our moduli space  $M$  can create, along with a “good” part containing the curves which behave as expected another part of higher dimension. One example of this is the Hilbert scheme of twisted cubics in  $P^3$ , which has expected dimension 12. We do get a nice 12-dimensional irreducible component containing twisted cubics, but there is also a 15-dimensional component containing schemes that are the union of plane cubics and a point [25].

Another example of how one might end up with a moduli space of higher dimension than expected comes from degree 2 maps from a genus 0 curve to a generic quintic threefold  $X$ . The moduli space of such maps has expected dimension 0, and there is a 0-dimensional component consisting of maps to the finite number of conics on  $X$ . However, in addition to this there is also a collection of one-dimensional families of maps, one for each line on  $X$ , representing double covers of this line.

There are typically two ways around these problems. The first is to prove that one can always deform  $X$  to another space  $X'$  such that the moduli space of curves on  $X'$  has the correct dimension. We can then calculate the numbers we want on  $X'$ , and check that they do not depend on which  $X'$  we deform to. In the setting of algebraic geometry such a strategy is not viable, as there are far too few deformations to have any hope of a general result stating that one can always deform  $X$  to a space with a moduli space of correct dimension.

Instead, one takes the second approach. The idea here is to use the deformation theory of the moduli problem to construct a homology class on the moduli space which has the dimension we want. We can then integrate against it to define the invariants, and check that the numbers we get are indeed invariant under deformations of  $X$ . This class is what is known as a virtual fundamental class.

## Motivating the construction

We give an example illustrating the idea of the construction of the virtual fundamental class in a concrete setting. The following way of motivating the virtual class is taken from [28, p.10]. The construction of the “virtual fundamental class” in this model case can be found in [12], sections 14.1 and 6.1.

We let  $Z$  be a scheme of dimension  $n$  with a rank  $r$  vector bundle  $E$  on it, and let  $s$  be a section of this bundle. We let  $M$ , which will eventually be our moduli scheme, be defined as the zero set  $Z(s)$  of the section. We may say that the virtual dimension of  $M$  is the dimension it would have if  $s$  were a transverse section, which in this case will be  $n - r$ . In case  $s$  is transverse, the pushforward of the fundamental class  $[M]$  is  $e(E) \cdot [Z]$ , where  $e(E)$  denotes the euler class of  $E$ .

If  $s$  is not transverse, we can still construct a class  $[M]^{\text{vir}}$  of the correct dimension, i.e. lying in  $A_{n-r}(M)$ . Furthermore, pushing forward this class to  $A_*(Z)$  gives the answer we would expect from the case where  $s$  is transverse, that is  $e(E) \cdot [Z]$ .

The construction of this class goes as follows. In the bundle  $E$  we consider a deformation of the graph of the section  $s$ , parametrized by  $\lambda$  and given by  $\lambda \cdot s$ . Letting  $\lambda$  go to  $\infty$ , this gives a cone in  $E|_M$ , informally speaking this is  $s$  made vertical. This cone can then be intersected with  $M$  inside  $E|_M$ , and the result is the correct cycle in  $A_*(M)$ .

The above construction is not applicable to more general moduli problems, as the setup is too restrictive to allow the moduli space to be written as  $Z(s)$  as above. However, the idea is that the construction of the class above was essentially done on  $M$ , and could therefore be carried out even without having an ambient space. From the above construction we really

only need the infinitesimal data on  $M$

$$0 \rightarrow T_M \rightarrow T_Z|_M \xrightarrow{ds} E|_M \rightarrow \text{ob} \rightarrow 0$$

where  $\text{ob}$  is some sheaf that can be thought of as containing obstructions to  $M$  being cut out by a transverse section of  $E$ .

Now, for a general moduli space  $M$  we have the tangent sheaf on  $M$ , and in many cases an obstruction sheaf  $\text{ob}$  naturally arising from the deformation theory of the moduli problem. The virtual dimension of the moduli problem is the dimension of  $T_M$  over a point  $p$  in  $M$  minus that of  $\text{ob}$  over  $p$ . Unless  $M$  is smooth, these dimensions differ for different  $p$ , but in the cases where a virtual fundamental class can be constructed the difference is constant, so the virtual dimension is well defined. What is needed for the construction of a virtual fundamental class is a two-term locally free resolution of these sheaves, which is to say an exact sequence

$$0 \rightarrow T_M \rightarrow E_0 \rightarrow E_1 \rightarrow \text{ob} \rightarrow 0,$$

where  $E_0$  and  $E_1$  are vector bundles. Here  $E_0$  and  $E_1$  play the roles of  $T_Z|_M$  and  $E|_M$  in the above example, respectively.

Given such a resolution the virtual fundamental class is constructed in [5]. Briefly, the construction goes by defining a suitable cone inside  $E_1$ , which is then intersected with  $M$ , giving the class  $[M]^{\text{vir}}$  that we want.

## Obstruction theory

We give the definition of a perfect obstruction theory on a scheme  $M$ . It is included here mostly as a reference point for some later invariant calculations that directly involve the obstruction theory. From [15] we get the following:

**Definition 1.** *A perfect obstruction theory consists of the following data:*

1. *A two term complex of vector bundles  $E^\bullet = [E^{-1} \rightarrow E^0]$  on  $M$ .*
2. *A morphism  $\phi$  in the derived category (of quasi-coherent sheaf complexes bounded from above) from  $E^\bullet$  to the cotangent complex  $L^\bullet M$  of  $M$  satisfying the following properties.*
  - (a)  *$\phi$  induces an isomorphism in cohomology degree 0*
  - (b)  *$\phi$  induces a surjection in cohomology in degree -1.*

Although we will not mention the cotangent complex  $L^\bullet M$  again, the definition and basic properties can be found in [11], p.226. We note that there is an isomorphism with the cotangent sheaf  $h^0(L_M^\bullet) \cong \Omega_M$ , so that by property (a) an obstruction theory  $E^\bullet$  will satisfy  $h^0(E^\bullet) \cong \Omega_M$ .

It is common to abuse notation by referring to the complex  $E^\bullet$  as the obstruction theory, suppressing mention of the map  $\phi$ .

We note that the virtual dimension  $vd$  of  $M$ , and hence the dimension of the virtual fundamental class, is the rank of the obstruction theory used to define it. That is, if  $E^\bullet = E^{-1} \rightarrow E^0$  is a complex of vector bundles defining an obstruction theory, the dimension of the virtual fundamental class defined from it will be

$$\dim[M]^{\text{vir}} = \text{rk} E^\bullet = \text{rk} E^0 - \text{rk} E^{-1}.$$

We denote the dual complex to  $E^\bullet$  by

$$E_\bullet = E_0 \rightarrow E_1.$$

The *obstruction sheaf* of the obstruction theory is the cokernel of this dual complex, i.e. we have

$$\text{ob} = h^1(E_\bullet).$$

This gives an alternative expression for the virtual dimension of  $M$ . For every point  $p$  in  $X$  we have

$$\begin{aligned} vd &= \text{rk} E_p^0 - \text{rk} E_p^1 = \text{rk} h^0(E^\bullet)_p - \text{rk} h^{-1}(E^\bullet)_p \\ &= \text{rk} h^0(E^\bullet)_p - \text{rk} h^1(E_\bullet)_p = \text{rk} \Omega_{M,p} - \text{rk} (\text{ob})_p, \end{aligned}$$

so the virtual dimension is equal to the dimension of the tangent space of  $M$  in  $p$  minus the dimension of the space of obstructions in  $p$ .

There is a quite explicit formula for the virtual fundamental class, namely, it is the part of

$$c(E_1 - E_0) \cap [c_F(M)],$$

living in the virtual dimension part of  $A_*(M)$  [26, Theorem 4.6]. Here  $c_F(M)$  is Fulton's total Chern class of the scheme  $M$ , see [12, 4.2.6].

In two special cases the virtual fundamental class is easily described (see [29, Chap. 26] for a further discussion of these in the Gromov-Witten setting). If the moduli problem is *unobstructed*, i.e. if  $\text{ob} = 0$ , the virtual fundamental class  $[M]^{\text{vir}}$  is equal to the usual fundamental class  $[M]$ . This happens for instance in the case of Gromov-Witten invariants counting maps of genus 0 curves to projective space, where the obstruction theory on  $\overline{M}_{0,r}(\mathbb{P}^n, \beta)$  gives a vanishing obstruction bundle.

The second easy case is that where the moduli space  $M$  is nonsingular. In this case the obstruction sheaf is in fact a bundle, and the virtual fundamental class is then the Euler class of  $\text{ob}$  times the usual fundamental class, i. e.

$$[M]^{\text{vir}} = e(\text{ob}) \cdot [M]. \quad (1.1)$$

One example where this applies is the Hilbert scheme of one, two or three points on a threefold, which occurs in Donaldson-Thomas theory. In the chapter on elliptic fibrations we shall see other examples of smooth Hilbert schemes of curves, arising from an isomorphism with the Hilbert scheme of points on a surface, which is known to be smooth.

A simple example demonstrating both of the above properties is the Hilbert scheme of lines in projective space,  $I_1(\mathbb{P}^3, L)$ , which is a Grassmannian of dimension 4. By Proposition 1 in section 1.2 the virtual dimension is given by the formula  $-K_{\mathbb{P}^3} \cdot L$ , and so is also 4. As the moduli space is smooth the obstruction sheaf is a bundle, and as the rank of  $E_\bullet$  must be 4, we get

$$\text{rk ob} = \text{rk } h^1(E_\bullet) = \text{rk } h^0(E_\bullet) - \text{rk } E_\bullet = \text{rk } T_X - 4 = 0.$$

Hence the moduli space is unobstructed and the virtual fundamental class is the ordinary fundamental class.

### Defining invariants from a virtual fundamental class

Equipped with a virtual fundamental class of the correct dimension we may now define invariants from this class. If this virtual dimension is zero, we simply integrate the fundamental class over the moduli space to obtain a number. This number will be the invariant, which we denote

$$\#^{\text{vir}}(M) = \int_{[M]^{\text{vir}}} 1.$$

More generally, if the virtual dimension is greater than zero, we first pick cohomology classes  $\gamma_i \in H^*(X)$  informally representing restrictions on the curves to be counted, i.e. we count only the curves meeting all of the classes  $\gamma_i$ . These  $\gamma_i$  can then be lifted to cohomology classes on the moduli space, and intersecting the virtual fundamental class with the lifted classes we get a zero-dimensional class on the moduli space. We then define the invariants to be the degree of this class. In this way we get more general invariants of the threefold, dependent on a choice of cohomology classes  $\gamma_i$ .

## 1.2 Donaldson-Thomas invariants

Getting somewhat more concrete, we begin with an outline of the definition of Donaldson-Thomas invariants. The moduli spaces will in this setting be Hilbert schemes parametrizing subschemes of  $X$  of dimension at most one, i.e. curves and points. Given such a subscheme  $Z$ , we let  $Z'$  be the largest purely one-dimensional subscheme of  $Z$ , that is  $Z$  with any isolated or embedded points removed. The curve class of  $Z$  is then defined as to be the fundamental class of this one-dimensional component,  $[Z'] \in H_2(X; \mathbb{Z})$ .

Specifying a class  $\beta \in H_2(X; \mathbb{Z})$  and an integer  $n$ , we denote by

$$I_n(X, \beta)$$

the Hilbert scheme parametrizing schemes  $Z$  with curve class  $\beta$  and such that the Euler characteristic of the structure sheaf  $\chi(\mathcal{O}_Z)$  is  $n$ . Note the special

case where  $\beta = 0$ , where we have  $I_n(X, 0) = X^{[n]}$ , the Hilbert scheme of  $n$  points on  $X$ .

There is an alternative way of describing this scheme, which we discuss briefly. There is a coarse moduli space parametrizing Gieseker semistable sheaves on  $X$  with given rank, Chern classes and determinant. See [17] for the definition of semistable as well as the existence of this moduli space (Theorem 4.3.4).

We may consider the special case of the moduli space  $M$  of rank 1 semistable sheaves on  $X$  with Chern classes  $(c_0, c_1, c_2, c_3) = (1, 0, -\beta, n)$  and trivial determinant. It can be shown that semistable rank 1 sheaves with trivial determinant naturally inject into  $\mathcal{O}_X$ , which is the same as saying that they are ideal sheaves on  $X$ . Therefore, to each sheaf in  $M$  there is an associated closed subscheme of  $X$ , which by the choice of Chern classes will lie in  $I_n(X, \beta)$ . It can be shown that this correspondence gives an isomorphism between  $M$  and  $I_n(X, \beta)$ .

In [28] Thomas uses the deformation theory of sheaves to construct a perfect obstruction theory on such a moduli space of sheaves, assuming that the anticanonical divisor of  $X$  is effective, plus some other mild hypothesis which is satisfied in the case we are interested in, see [28, Corollary 3.39]. If we restrict to the case of appropriate Chern classes and determinant, by the isomorphism above this gives a perfect obstruction theory on  $I_n(X, \beta)$ , and hence a virtual fundamental class  $[I_n(X, \beta)]^{\text{vir}}$ .

**Proposition 1.** *The virtual dimension of  $I_n(X, \beta)$  with this obstruction theory is  $-K_X \cdot \beta$ .*

*Proof.* See [22, Lemma 1]. □

**Definition 2.** *Let  $X$  be a smooth, projective threefold with an effective anticanonical divisor. We define the Donaldson-Thomas invariant of  $X$  with respect to  $\beta$  and  $n$  to be*

$$D_{n,\beta} = \#^{\text{vir}}(I_n(X, \beta)) = \int_{[I_n(X, \beta)]^{\text{vir}}} 1.$$

For general  $X$  and  $\beta$  this number will be trivially 0, as the virtual dimension of the Hilbert scheme is likely to be nonzero. As mentioned previously, this can be remedied by intersecting  $[I_n(X, \beta)]^{\text{vir}}$  with cohomology classes of appropriate dimension to get a 0-dimensional class. We will see how this is done in the Donaldson-Thomas setting later. We now consider some natural classes of examples of cases that do give virtual dimension 0.

The first example we consider is when  $X$  has trivial canonical divisor. Here it can be shown that the obstruction sheaf  $\text{ob}$  on  $I_n(X, \beta)$  is isomorphic to the cotangent sheaf  $\Omega_{I_n(X, \beta)}$ . If  $I_n(X, \beta)$  happens to be smooth of

dimension  $d$ , by using (1.1) the Donaldson-Thomas invariant reduces up to sign to the Euler characteristic of  $I_n(X, \beta)$ .

$$\begin{aligned} D_{n,\beta} &= \int_{[I_n(X,\beta)]^{\text{vir}}} 1 = \int_{[I_n(X,\beta)]} e(\text{ob}) = \int_{[I_n(X,\beta)]} e(\Omega_{I_n(X,\beta)}) \\ &= \int_{[I_n(X,\beta)]} (-1)^d e(T_{I_n(X,\beta)}) = (-1)^d \chi(I_n(X, \beta)). \end{aligned}$$

Another case in which Donaldson-Thomas invariants coincide with Euler characteristics is when  $X$  is a Calabi-Yau threefold and  $\beta = 0$ , so the moduli space is the Hilbert scheme of points on  $X$ . Here the formula

$$D_{n,0} = (-1)^n \chi(I_n(X, 0)) \quad (1.2)$$

has been shown to hold for all  $n$ . In case  $n = 1, 2, 3$  the space  $I_n(X, 0)$  is smooth so this is just a consequence of the previous paragraph. For bigger  $n$  different methods are needed. Several proofs of (1.2) exist, see for example [6].

When the moduli space is not smooth or  $X$  is not Calabi-Yau, the Donaldson-Thomas invariants are in general different from the Euler characteristic of the moduli space. For example we may consider the simplest non-trivial Donaldson-Thomas invariant,  $D_{1,0}$ , where the moduli space  $I_1(X, 0)$  is isomorphic to the threefold  $X$  itself. Here it can be shown ([21, Lemma 3]) that the obstruction sheaf is

$$\text{ob} \cong (T_X \otimes K_X)^\vee.$$

Using the fact that  $I_1(X, 0) \cong X$  is smooth, we get

$$D_{1,0} = - \int_{[X]} e(T_X \otimes K_X) = \int_{[X]} c_1(T_X)c_2(T_X) - c_3(T_X).$$

Taking  $X = \mathbb{P}^3$ , for example, we get  $D_{1,0} = 20$ , in contrast to  $\chi(\mathbb{P}^3) = 4$ .

### Donaldson-Thomas partition functions

Working with Donaldson-Thomas invariants it is convenient to gather all the invariants in a power series, known as the Donaldson-Thomas partition function:

$$Z_{DT}(X; q, v) = \sum_{\beta \in H_2(X; \mathbb{Z})} \sum_{n \in \mathbb{Z}} D_{n,\beta} q^n v^\beta.$$

The case  $\beta = 0$  plays a special role in the theory, and we collect the invariants for the degree 0 case in

$$Z_{DT}(X, \beta)_0 = \sum_{n \in \mathbb{Z}} D_{n,0} q^n.$$



In both curve counting theories we have a notion of reduced invariants, which informally is what you get from disregarding the contribution from objects in the moduli space that have components representing points. In Gromov-Witten theory this can be achieved geometrically, as there is a natural moduli space of curve maps which do not collapse any connected component of the curve to a point in  $X$ , so that the image of the curve is purely 1-dimensional.

This is not an option in Donaldson-Thomas theory, because the ideal sheaves in  $I_n(X, \beta)$  corresponding to subschemes  $Z \subset X$  with no zero-dimensional component do not form a proper subscheme. Instead we remove the contribution of points by a formal method. This definition mirrors the algebraic relation between reduced and nonreduced partition functions of Gromov-Witten invariants. We define the *reduced* Donaldson-Thomas partition function of  $X$  by

$$Z'_{DT}(X; q, v) = Z_{DT}(X; q, v) / Z_{DT}(X; q)_0. \quad (1.3)$$

As a final definition in this section, we isolate the reduced invariants coming from a curve class  $\beta$  in a single power series, defined by

$$Z'_{DT}(X; q, v) = 1 + \sum_{\beta \neq 0} Z'_{DT}(X; q)_\beta q^n v^\beta.$$

### Donaldson-Thomas invariants with insertions

We shall now deal with the more general case of Donaldson-Thomas invariants dependent on cohomology classes  $\gamma_i$  on  $X$ . These invariants are sometimes called descendent invariants, by virtue of having some connection with so-called descendent fields in physics. They were first defined in [22].

Consider the space  $I_n(X, \beta) \times X$  and let  $\pi_1$  and  $\pi_2$  be the projections to the first and second factors, respectively. As  $I_n(X, \beta)$  is a fine moduli space, we have the universal ideal sheaf  $\mathcal{I}$  defined on  $I_n(X, \beta) \times X$ .

For  $\gamma \in H^l(X, \mathbb{Z})$ , we let  $\text{ch}_{k+2}(\gamma_i)$  denote the following operation on the homology of  $I_n(X, \beta)$ :

$$\begin{aligned} \text{ch}_{k+2}(\gamma) : H_*(I_n(X, \beta); \mathbb{Q}) &\rightarrow H_{*-l-2k+2}(I_n(X, \beta); \mathbb{Q}), \\ \text{ch}_{k+2}(\gamma)(\xi) &= \pi_{1*}(\text{Ch}_{k+2}(\mathcal{I}) \cdot \pi_2^*(\gamma) \cap \pi_1^*(\xi)). \end{aligned}$$

Here  $\text{Ch}_{k+2}(\mathcal{I})$  denotes the  $k+2$ -th term of the Chern character of  $\mathcal{I}$ .

We let  $\tilde{\tau}_k(\gamma)$  correspond to the operation  $(-1)^{k+1} \text{ch}_{k+2}(\gamma)$ . We choose  $r$  cohomology classes  $\gamma_1, \dots, \gamma_r \in H^*(X, \mathbb{Z})$ , and  $r$  integers  $k_1, \dots, k_r$ . We then have

**Definition 3.** *Let  $X$  be a smooth projective threefold with effective anti-canonical divisor. We define the Donaldson-Thomas invariants of  $X$ , depending on  $\beta \in H_2(X; \mathbb{Z})$ ,  $\gamma_1, \dots, \gamma_r \in H^*(X; \mathbb{Z})$  and  $n, k_1, \dots, k_r \in \mathbb{Z}$ , to*

be

$$\langle \tilde{\tau}_{k_1}(\gamma_1) \cdots \tilde{\tau}_{k_r}(\gamma_r) \rangle_{n,\beta} = \int_{[I_n(X,\beta)]^{\text{vir}}} \prod (-1)^{k_i+1} \text{ch}_{k_i+2}(\gamma_i).$$

The integral above is to be interpreted as the push-forward to a point of the class

$$(-1)^{k_1+1} \text{ch}_{k_1+2}(\gamma_1) \circ \cdots \circ (-1)^{k_r+2} \text{ch}_{k_r+1}(\gamma_r) ([I_n(X,\beta)]^{\text{vir}}).$$

Note that taking  $r$  to be zero we get Definition 2 of the original invariants. In contrast with ordinary Donaldson-Thomas invariants, the descendent invariants may take rational instead of integer values, on account of the fact that the Chern character has rational coefficients.

The meaning of these invariants is easiest to see where the  $k_i$  are all equal to 0. This is also known as the case of invariants with *primary insertions* or primary fields for physics reasons. The operation  $\tilde{\tau}_0(\gamma)$  roughly represents imposing the condition that the curve  $[Z] \in I_n(X,\beta)$  meet  $\gamma$ .

An illustration of this is the result from [10] stating that given  $\gamma \in H^2(X;\mathbb{Z})$  and  $\xi \in H_*(I_n(X,\beta);\mathbb{Q})$  such that  $\xi$  can be represented by an algebraic cycle, we have

$$\tilde{\tau}_0(\gamma)(\xi) = \int_{\beta} \gamma \cdot \xi.$$

This is what one would expect from imposing the enumerative condition that curves of class  $\beta$  meet a cohomology class  $\gamma$ . We note that the equation above is also expected to hold for more general  $\xi$ , i.e. for those not necessarily represented by an algebraic cycle.

We may also note that taking the invariants to have primary insertions  $\langle \tilde{\tau}_0(\gamma_1) \cdots \tilde{\tau}_0(\gamma_r) \rangle_{n,\beta}$  we get integer values. This can be seen as follows. Let  $Z \subset I_n(X,\beta) \times X$  be the universal closed subscheme. We then have

$$\text{Ch}_2(\mathcal{I}) = -\text{Ch}_2(\mathcal{O}_Z) = -c_2(\mathcal{O}_Z) + \frac{c_1(\mathcal{O}_Z)}{2} = -c_2(\mathcal{O}_Z),$$

where the last equality holds because the support of  $\mathcal{O}_Z$  has codimension 2, implying that  $c_1(\mathcal{O}_Z) = 0$ . Thus no denominators occur in the definition of  $\tilde{\tau}_0(\gamma)$ , giving the integrality of  $\langle \tilde{\tau}_0(\gamma_1) \cdots \tilde{\tau}_0(\gamma_r) \rangle_{n,\beta}$ .

Finally we mention that as in the case of the Donaldson-Thomas invariants without insertions, the general DT-invariants can be collected in a partition function

$$Z_{DT} \left( X; q \mid \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_i) \right)_{\beta} = \sum_{n \in \mathbb{Z}} \langle \tilde{\tau}_{k_1}(\gamma_1) \cdots \tilde{\tau}_{k_r}(\gamma_r) \rangle_{n,\beta} q^n.$$

There is also a reduced partition function, given by

$$Z'_{DT} \left( X; q \mid \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_i) \right)_{\beta} = \frac{Z_{DT}(X; q \mid \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_i))_{\beta}}{Z_{DT}(X; q)_0}$$

### 1.3 Gromov-Witten invariants

Having seen one way of parametrizing and counting curves on  $X$ , we now turn to Gromov-Witten theory, where we use a different notion of a curve on  $X$ , namely a map from a curve to  $X$ . Ideally, we could want to form the moduli space of isomorphism classes of maps from a smooth curve of genus  $g$  to  $X$ , with the natural definitions of what constitutes a family of such maps and isomorphisms between families.

There are a few problems with this naive approach. First of all, such a space is likely to be non-proper. This defect can be amended by allowing the source curves to be reducible and to have nodal singularities. Secondly, the curve maps can turn out to have non-trivial automorphisms, which makes it impossible to define universal families over the spaces. The solution to this problem is first to add to the notion of a curve map  $\mu : C \rightarrow X$  the data of  $r$  marked points on the curve  $C$ , and secondly restricting to those resulting objects that have finite automorphism group. The possible existence of nontrivial automorphisms still excludes the possibility of finding a moduli scheme, but as automorphism groups are finite it is possible to construct a Deligne-Mumford moduli stack  $\overline{M}_{g,r}(X, \beta)$  parametrizing maps of curves to  $X$  [7, Theorem 3.14].

The added data of  $r$  marked points on a curve also allows us to define more general invariants depending on  $r$  cohomology classes in  $H^*(X)$  which informally represent conditions on the curves to be counted.

We give the precise definition of the notions involved in defining Gromov-Witten invariants, following the presentation in [13].

An  $r$ -pointed, genus  $g$ , *quasi-stable* curve  $(C, p_1, \dots, p_r)$  is a projective, connected, reduced curve with at most nodal singularities, together with  $r$  distinct, nonsingular points of the curve. A family of quasi-stable curves over a scheme  $S$  is a flat projective map  $\pi : \mathcal{C} \rightarrow S$  together with sections  $p_i : S \rightarrow \mathcal{C}$  such that each geometric fibre  $(\mathcal{C}_s, p_1(s), \dots, p_r(s))$  is a quasi-stable curve.

Given a scheme  $X$  a family of maps from  $r$ -pointed, genus  $g$  curves to  $X$  consists of the data  $(\pi : \mathcal{C} \rightarrow S, \{p_i\}_{i=1}^r, \mu : \mathcal{C} \rightarrow X)$ , where  $(\pi : \mathcal{C} \rightarrow S, \{p_i\}_{i=1}^r)$  is a family of  $r$ -pointed, genus  $g$ , quasi-stable curves, and  $\mu$  is any morphism.

An isomorphism of two families of maps over  $S$ ,

$$(\pi : \mathcal{C} \rightarrow S, \{p_i\}_{i=1}^r, \mu : \mathcal{C} \rightarrow X), \quad (\pi' : \mathcal{C}' \rightarrow S, \{p'_i\}_{i=1}^r, \mu' : \mathcal{C}' \rightarrow X)$$

is an isomorphism of schemes  $\gamma : \mathcal{C} \rightarrow \mathcal{C}'$  such that  $\pi = \pi' \circ \gamma$ ,  $p'_i = \gamma \circ p_i$  and  $\mu = \mu' \circ \mu$ .

For every irreducible component  $E$  of a quasi-stable curve  $C$  we let the *special points* of  $E$  be the ones which are either intersections of different components or marked points. We say a map  $\mu : C \rightarrow X$  is *stable* if the following two conditions hold, for every irreducible component  $E$  of  $C$ :

1. If  $E \cong \mathbb{P}^1$  and  $E$  is mapped to a point, there are at least 3 special points on  $E$ .
2. If  $E$  has arithmetic genus 1 and is mapped to a point, there is at least one special point on  $E$ .

An automorphism of an  $r$ -pointed curve map  $(C, \{p_i\}, \mu)$  is an automorphism  $\gamma$  of  $C$  such that  $\gamma(p_i) = p_i$  for all  $i$  and  $\gamma \circ \mu = \mu$ . It is easily checked that the two conditions for stability are equivalent to the curve map having finite automorphism group. This finiteness is needed in order to get a good moduli space of maps.

A family of stable maps of  $r$ -pointed, genus  $g$  curves is a family of maps such that each geometric fibre in the family is stable. Let  $\beta \in H_2(X; \mathbb{Z})$ . We say a map of a curve  $\mu : C \rightarrow X$  *represents*  $\beta$  if the pushforward of the fundamental class of  $C$  is  $\beta$ . We now arrive at the definition of our moduli functor, which is the contravariant functor from the category of complex algebraic schemes to sets sending a scheme  $S$  to the set of isomorphism classes of families over  $S$  of stable maps of  $r$ -pointed, genus  $g$  curves to  $X$  representing  $\beta$ . There is a proper Deligne-Mumford stack  $\overline{M}_{g,r}(X, \beta)$  which is a moduli space for this functor.

As in the case of the Hilbert scheme, there exists a perfect obstruction theory on  $\overline{M}_{g,r}(X, \beta)$ , which allows for the construction of a virtual fundamental class  $[\overline{M}_{g,r}(X, \beta)]^{\text{vir}}$  in the Chow group of  $\overline{M}_{g,r}(X, \beta)$ .

Note that all of the definitions and constructions above apply to a general smooth scheme  $X$ , not necessarily of dimension 3. As a simple example, taking  $X$  to be a point we get

$$\overline{M}_{g,r}(\text{pt}, 0) \cong \overline{M}_{g,r},$$

where  $\overline{M}_{g,r}$  is a moduli space parametrizing stable curves with  $r$  marked points, that is quasistable curves with finite automorphism group.

In the case where  $X$  is a threefold, the virtual dimension of  $\overline{M}_{g,r}(X, \beta)$  and hence the dimension of this virtual fundamental class is  $-K_X \cdot \beta + r$ .

**Definition 4.** *Let  $X$  be a projective, smooth threefold. The Gromov-Witten invariant of  $X$  with respect to  $n$  and  $\beta$  is*

$$N_{g,\beta} = \int_{[\overline{M}_{g,0}(X, \beta)]^{\text{vir}}} 1.$$

As a consequence of the fact that  $\overline{M}_{g,0}(X, \beta)$  is a stack, this number may be rational, as opposed to the Donaldson-Thomas invariant  $D_{n,\beta}$ , which is always an integer.

As with Donaldson-Thomas invariants, we may collect all the Gromov-Witten invariants of  $X$  in power series. We define the reduced Gromov-Witten potential to be the series

$$F'_{GW}(X; u, v) = \sum_{\beta \neq 0} \sum_{g \geq 0} N_{g,\beta} u^{2g-2} v^\beta.$$

The reduced partition function is then

$$Z'_{GW}(X; u, v) = \exp F'_{GW}(X; u, v).$$

This series can alternatively be defined as the generating function for Gromov-Witten invariants defined from moduli spaces  $\overline{M}'_{g,r}(X, \beta)$  of stable maps of curves where the source curve is possibly disconnected and no connected component is mapped to a point.

We let  $Z'_{GW}(X; u)_\beta$  denote the reduced degree  $\beta$  partition function,

$$Z'_{GW}(X; u, v) = 1 + \sum_{\beta \neq 0} Z'_{GW}(X; u)_\beta v^\beta.$$

Notice that we do not include the terms where  $\beta = 0$ , i.e. the moduli space of curves mapping to a point. This is the reason for the qualifier “reduced” for the partition functions and the primes. We may alternatively first define the unreduced potential

$$F_{GW}(X; u, v) = \sum_{\beta} \sum_{g \geq 0} N_{g,\beta} u^{2g-2} v^\beta$$

giving the unreduced partition function  $Z_{GW}(X; u, v)$ . Then define the degree 0 potential by

$$F_{GW}(X; u)_0 = \sum_{g \geq 0} N_{g,0} u^{2g-2},$$

giving the degree 0 partition function  $Z_{GW}(X; u)_0$ . The reduced partition function is then obtained by taking

$$Z'_{GW}(X; u, v) = Z_{GW}(X; u, v) / Z_{GW}(X; u)_0.$$

This is precisely the algebraic relation (1.3) used to define the reduced Donaldson-Thomas partition functions  $Z'_{DT}(X; u, q)$ .

### Gromov-Witten invariants with insertions

More general Gromov-Witten invariants may be defined by considering the space  $\overline{M}_{g,r}(X, \beta)$  with  $r > 0$ . There are evaluation maps  $\text{ev}_i$  on  $\overline{M}_{g,r}(X, \beta)$

$$\text{ev}_i : \overline{M}_{g,r}(X, \beta) \rightarrow X$$

sending a point representing a curve map  $\mu : C \rightarrow X$  to the image of the  $i$ -th marked point  $\mu(p_i)$  of the curve. We use these to define so-called Gromov-Witten invariants with insertions.

**Definition 5.** Given a threefold  $X$ , we define Gromov-Witten invariants  $\langle \tau_0(\gamma_1) \cdots \tau_0(\gamma_r) \rangle_{g,\beta}$  depending on  $\beta \in H_2(X; \mathbb{Z})$ ,  $g \in \mathbb{Z}$  and  $\gamma_1, \dots, \gamma_r \in H^*(X; \mathbb{Z})$  by

$$\langle \tau_0(\gamma_1) \cdots \tau_0(\gamma_r) \rangle_{g,\beta} = \int_{[\overline{M}_{g,r}(X,\beta)]^{\text{vir}}} \text{ev}_1^*(\gamma_1) \cup \cdots \cup \text{ev}_r^*(\gamma_r).$$

In the cases where Gromov-Witten invariants correspond with classical enumerative counts, this definition corresponds to imposing conditions on the curves to be counted by the cohomology classes  $\gamma_i$ . Note also that taking  $r = 0$  gives the previous definition of Gromov-Witten invariants without intersection with cohomology classes  $\gamma_i$ .

The seemingly superfluous  $\tau_0$ 's in the notation for Gromov-Witten invariants appear because there exist still more general invariants, known as descendent invariants, defined via homology operations  $\tau_k(\gamma)$  and denoted  $\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) \rangle_{g,\beta}$ . We will not be using or seeing more of these descendant invariants.

As usual we may collect the invariants in generating functions. We have the reduced Gromov-Witten potential

$$F'_{GW} \left( X; u, v \mid \prod_{i=1}^r \tau_0(\gamma_i) \right) = \sum_{\beta \neq 0} \sum_{g \geq 0} \langle \tau_0(\gamma_1) \cdots \tau_0(\gamma_r) \rangle_{g,\beta} u^{2g-2} v^\beta.$$

For  $\beta \neq 0$ , we get a reduced Gromov-Witten partition function

$$Z'_{GW} \left( X; u \mid \prod_{i=1}^r \tau_0(\gamma_i) \right)_\beta$$

defined by

$$1 + \sum_{\beta \neq 0} Z'_{GW} \left( X; u \mid \prod_{i=1}^r \tau_0(\gamma_i) \right)_\beta v^\beta = \exp F'_{GW} \left( X; u, v \mid \prod_{i=1}^r \tau_0(\gamma_i) \right).$$

## 1.4 The MNOP conjectures

In [21] three conjectures were proposed regarding the Donaldson-Thomas invariants and a relation to Gromov-Witten invariants. The first of these describes the degree 0 part of Donaldson-Thomas theory.

**Conjecture 1.** *The degree 0 partition function is determined by*

$$Z_{DT}(X; q, v)_0 = M(-q)^{\int_{[X]} c_3(T_X \otimes K_X)}.$$

*In particular, if  $X$  is Calabi-Yau, we have*

$$Z_{DT}(X; q, v)_0 = M(-q)^{\chi(X)}.$$

Here  $M(q)$  is the MacMahon function, defined by

$$M(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-n}.$$

MacMahon proved this to be the generating series for the number of 3-dimensional partitions (see Definition 6) of size  $n$ . The MacMahon function turns up a lot in Donaldson-Thomas computations, we will see one natural way this happens in section 2.3.

Cheah proved in [8] that for a smooth threefold  $X$  the generating function for the Euler characteristic of the Hilbert scheme of  $n$  points on  $X$  is

$$\sum_{n \geq 0} \chi(X^{[n]}) q^n = M(q)^{\chi(X)}.$$

Hence in the case where  $X$  is Calabi-Yau the result above can be interpreted as saying that up to sign the virtual count of  $X^{[n]}$  equals the Euler characteristic.

**Conjecture 2.** *The reduced series  $Z_{DT}(X; q)_{\beta}$  is a rational function of  $q$ . If  $X$  is Calabi-Yau, this function is symmetric under the transformation  $q \mapsto 1/q$ .*

The symmetry under inversion of  $q$  when  $X$  is Calabi-Yau can be seen as a consequence of the fact that Gromov-Witten invariants are real. The Gromov-Witten series should be invariant under the substitution  $e^{iu} \mapsto e^{-iu}$ , and assuming the Donaldson-Thomas/Gromov-Witten correspondence below this makes the Donaldson-Thomas partition function invariant under  $q \mapsto 1/q$ .

The final conjecture relates the reduced Donaldson-Thomas partition function to the reduced Gromov-Witten partition function.

**Conjecture 3 (Donaldson-Thomas/Gromov-Witten correspondence).**

*Let  $d = -K_X \cdot \beta$ . Making the substitution  $q = -e^{iu}$ , we have*

$$(-iu)^d Z'_{GW}(X; u)_{\beta} = (-q)^{-d/2} Z'_{DT}(X; q)_{\beta}.$$

*In particular, if  $X$  is Calabi-Yau, after substituting  $q = -e^{iu}$  we get*

$$Z'_{GW}(X; u, v) = Z'_{DT}(X; q, v).$$

## Progress on the conjectures

Conjecture 1 on the degree 0 Donaldson-Thomas invariants is now a theorem, as is several of its generalizations to settings where  $X$  is not a proper threefold. Several proofs exist, see [6], [19] and [20]. Conjecture 2 and 3 are proved in special cases, for instance when  $X$  is toric [23], and the analogue conjectures for the case where  $X$  is a rank 2 bundle over a curve [24].





## Chapter 2

# Computation techniques

### 2.1 Equivariant theory

In order to explain the technique of toric localization, we give a brief introduction to equivariant cohomology, following Fulton's lectures [14] on equivariant cohomology in algebraic geometry. This will also be of use in one of the arguments in the calculations on trivial elliptic fibrations to be presented later.

The idea of equivariant cohomology is most easily formulated in the setting of algebraic topology. We begin with a Lie group  $G$  and a topological space  $X$  on which  $G$  acts on the left. We let  $EG$  be a contractible space on which  $G$  acts freely. Form the new space

$$EG \times^G X = EG \times X / (e \cdot g, x) \sim (e, g \cdot x).$$

We may then define the equivariant cohomology of  $X$  with respect to  $G$ , written as  $H_G^*(X)$ . It is defined as the usual singular cohomology of  $EG \times^G X$ , that is

$$H_G^*(X) = H^*(EG \times^G X).$$

It can be shown that this definition does not depend on the choice of contractible  $G$ -space  $EG$ .

The equivariant cohomology theory enjoys most of the properties of an ordinary cohomology theory, such as pullback and characteristic classes of (equivariant) vector bundles. We let  $BG = EG/G$ , otherwise known as the classifying space of  $G$ . The equivariant cohomology of a point is

$$H_G^*(\text{pt}) = H^*(EG \times^G \text{pt}) = H^*(BG).$$

We let  $\Lambda_G = H^*(BG)$ . From the map  $X \rightarrow \text{pt}$  we get a map

$$\Lambda_G = H^*(BG) \rightarrow H_G^*(X),$$

hence the  $G$ -equivariant cohomology ring of a space  $X$  has a canonical  $\Lambda_G$ -algebra structure.

If  $F$  is a  $G$ -equivariant complex vector bundle on  $X$ , there are equivariant Chern classes  $c_i^G(F) \in H_G^{2i}(X)$ , defined as follows: As  $F$  is equivariant, we get a vector bundle over  $EG \times^G X$  which is

$$EG \times^G F \rightarrow EG \times^G X.$$

We may then define the equivariant Chern class  $c_i^G(F)$  to be the ordinary  $i$ -th Chern class of this bundle, which is a class in  $H^{2i}(EG \times^G X)$ . Note that this equivariant Chern class depends on the  $G$ -equivariant structure of  $F$  in addition to the usual vector bundle properties. In particular, even if  $F$  is a vector bundle over a point, the Chern classes of  $F$  may be non-trivial.

### Example of $G$ being a torus

The most useful example for our purposes is the one where the group is a torus  $T \cong \mathbb{C}^*$ . In this case we get the contractible space

$$ET = \mathbb{C}^\infty \setminus \{0\},$$

while the classifying space is

$$BT = \mathbb{CP}^\infty,$$

the infinite dimensional complex projective space. This has cohomology ring

$$\Lambda_T = H^*(BT) = H^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[t],$$

where  $t$  is the first Chern class of the tautological line bundle  $\mathcal{O}(-1)$  on  $\mathbb{CP}^\infty$ .

More generally, if  $T$  is an  $n$ -dimensional torus  $T \cong (\mathbb{C}^*)^n$ , we get the classifying space

$$BT = (\mathbb{CP}^\infty)^n$$

and the cohomology ring

$$\Lambda_T = H^*((\mathbb{CP}^\infty)^n) = \mathbb{Z}[t_1, \dots, t_n].$$

### Equivariant intersection theory

The spaces  $EG$  and  $BG$  involved in the definition of equivariant cohomology are often far from being algebraic, as they are typically infinite-dimensional. Nevertheless, the existence of an algebraic analogue to equivariant cohomology, that is of equivariant Chow groups, has been shown in [9]. Briefly, the algebraic definition goes by finding finite-dimensional algebraic approximations  $EG_m \rightarrow BG_m$  for every integer  $m$  and taking the equivariant Chow group  $A_k^G(X)$  to be the Chow groups of

$$X \times^G EG_{m(k)},$$

for some  $m(k)$  chosen large enough. It can be checked that this equivariant Chow group is independent of the choice of approximation spaces. The resulting theory has many of the standard properties of ordinary Chow groups, such as Chern classes of equivariant bundles and proper push-forward.

### Localization techniques

The reason for introducing equivariant cohomology and intersection theory is to be able to describe an important technique for calculating Donaldson-Thomas invariants, namely that of localization. In general localization is a way of expressing equivariant classes on a scheme with a group action by classes on the fixed point locus of the group action. The localization formula appearing in the context of virtual fundamental classes is proved in the article [15]. It is similar to and in some sense a generalization of the classical Atiyah-Bott localization formula [2] in equivariant cohomology.

The basic setup is as follows. We have a moduli space  $Y$  with an action of an algebraic torus  $T = (\mathbb{C}^*)^n$  on it. There is a virtual obstruction theory defined on  $Y$  that is equivariant with respect to the action of  $T$ . From the  $T$ -action we get a closed subscheme  $Y^T \subset Y$ , defined as the largest closed subscheme of  $Y$  such that the restriction of the  $T$ -action is trivial.

We assume for simplicity that the virtual dimension of  $Y$  is 0, so that our goal is to evaluate the integral of the virtual fundamental class  $[Y]^{\text{vir}}$  over  $Y$ . Since the obstruction theory on  $Y$  is  $T$ -equivariant, we may define an equivariant virtual fundamental class in  $A_*^T(Y)$ , which we by abuse of notation also denote by  $[Y]^{\text{vir}}$ . We further assume that we may calculate the invariant we are after using this equivariant virtual class, by taking the image of  $[Y]^{\text{vir}}$  under the map

$$A_0^T(Y) \rightarrow A_0^T(\text{pt}) = \mathbb{Z}.$$

Let  $\iota : Y^T \rightarrow Y$  be the inclusion. The idea is to find a class  $[Y^T]'$  in the equivariant cohomology ring  $A_*^T(Y^T)$  such that we have  $\iota_*([Y^T]') = [Y]^{\text{vir}}$ . Finding such a class we may calculate our original integral  $\int [Y]^{\text{vir}}$  as the integral of  $[Y^T]'$  over  $Y^T$ . As  $Y^T$  in good cases is a disjoint union of  $T$ -fixed points, we have reduced our calculation to a sum of fixed point contributions. The bulk of the work in such a calculation is then to describe what the contributions from given fixed points are.

Precisely, the localization formula is

$$[Y]^{\text{vir}} = \iota_* \sum \frac{[Y_i]^{\text{vir}}}{e(N_i^{\text{vir}})}.$$

Here  $Y_i$  are the connected components of  $Y^T$ . Note that to be able to divide by  $e(N_i^{\text{vir}})$  we must add inverses to the Chow ring  $A_*^T(Y)$ , so the equation is to be interpreted as holding in the localized ring

$$A_*^T(Y) \otimes \mathbb{Q}(t_1, \dots, t_n),$$

where the tensor product is taken over  $\Lambda_T \cong \mathbb{Z}[t_1, \dots, t_n]$ .

The obstruction theory on each connected component  $Y_i$  comes from taking the original obstruction theory restricted to  $Y_i$

$$E^{-1}|_{Y_i} \rightarrow E^0|_{Y_i}$$

and considering the part of this that is fixed under the action of  $T$ .

The  $N_i^{\text{vir}}$  appearing in the denominator is the virtual normal bundle of  $Y_i$ , defined by first taking the complex

$$E_0 \rightarrow E_1$$

restricted to  $Y_i$ . We consider the *moving part* of this complex, that is in each bundle we take the maximal subbundle such that the action of  $T$  on the bundle has no fixed points, and so get a new complex  $N_i^{\text{vir}}$

$$E_0|_{Y_i}^m \rightarrow E_1|_{Y_i}^m.$$

The Euler class is defined “ $K$ -theoretically”, so the Euler class of a complex  $[B_0 \rightarrow B_1]$  is taken to be  $e(B_0)/e(B_1)$ .

In the case of Donaldson-Thomas theory the  $T$ -action on the moduli space  $Y$  typically stems from an action of  $T$  on the threefold  $X$ . Such an action naturally induces a  $T$ -action on the Hilbert scheme  $I_n(X, \beta)$ . This  $T$ -action also induces a  $T$ -equivariant structure on the obstruction theory of  $I_n(X, \beta)$ .

Likewise, in Gromov-Witten theory a  $T$ -action on  $X$  gives a natural  $T$ -action on the space  $\overline{M}_{g,r}(X, \beta)$ . The method of localization in Gromov-Witten theory was first used by Kontsevich in [18].

The localization formula above may be used to define invariants in settings where the moduli space is nonproper if the fixed point scheme is proper. One such case is in the local Donaldson-Thomas theory of curves treated in [24], which we now present.

## 2.2 The local Donaldson-Thomas theory of curves

In this variant of Donaldson-Thomas theory, the three-dimensional scheme on which one counts curves is a rank 2 vector bundle over a smooth, proper curve  $C$ . In this particular setting we shall denote the threefold  $N$  rather than the usual  $X$ , following the notation in [24], from which the material presented here is gathered.

The Hilbert scheme in question is denoted  $I_n(N, d)$ , parametrizing proper subschemes  $Z \subset N$  of dimension not greater than 1. As usual,  $n$  denotes the Euler characteristic  $\chi(\mathcal{O}_Z)$ . The integer  $d$  is an analogue of specifying the curve class  $\beta \in H_2(N; \mathbb{Z})$  in the normal Donaldson-Thomas theory. We define  $d$  as the length of the intersection

$$Z \cap N_p$$

where  $N_p$  is the fibre over a generic point  $p$  of  $C$ .

As  $N$  is not projective but only quasi-projective, the Hilbert scheme  $I_n(N, d)$  is not proper. Hence we cannot integrate the virtual fundamental class as in the definition of Donaldson-Thomas, but we may still define Donaldson-Thomas invariants using localization techniques. For convenience, we assume that  $N$  is isomorphic to the direct sum of two line bundles over  $C$ . There is then an action of a two-dimensional torus  $T$  on  $N$ , defined by taking the direct sum of the scaling action of  $\mathbb{C}^*$  on each of the line bundles. If  $N$  is not decomposable in this manner, we can still define the equivariant Donaldson-Thomas invariants with respect to the scaling action of  $\mathbb{C}^*$  on  $N$ . However, any rank two bundle is deformation equivalent to a split bundle over  $C$ . Thus we can obtain the invariants of the indecomposable case by deforming to a split bundle and restricting to the diagonal torus.

Ideally, we would define the Donaldson-Thomas invariants of  $N$  as

$$\int_{[I_n(N, d)]^{\text{vir}}} 1,$$

but because  $I_n(N, d)$  is not proper this does not make sense. Instead, by considering the virtual localization formula we see that a sensible definition for the Donaldson-Thomas invariants of  $I_n(N, d)$  would be

$$\int_{[I_n(N, d)^T]^{\text{vir}}} \frac{1}{e(\text{Norm}^{\text{vir}})}. \quad (2.1)$$

Here  $\text{Norm}^{\text{vir}}$  is the virtual normal bundle of the embedding

$$I_n(N, d)^T \rightarrow I_n(N, d),$$

and the Euler class is the equivariant one. The integral is defined by taking the pushforward to a point, hence the invariants in this case take values in the equivariant cohomology ring of a point, suitably localized to accomodate the denominator in (2.1):

$$H_T^*(\text{pt})_{t_1, t_2} \cong \mathbb{Q}[t_1, t_2, t_1^{-1}, t_2^{-1}].$$

As in the case of absolute Donaldson-Thomas invariants, we may collect the invariants for different values of  $n$  in one generating function. We fix  $d$ , and let

$$Z(N)_d = \sum_{n \in \mathbb{Z}} \int_{[I_n(N, d)]^{\text{vir}}} \frac{1}{e(\text{Norm})^{\text{vir}}} q^n$$

As before, we wish to disregard the contribution of degree 0 invariants, so we form the reduced series

$$Z'(N)_d = Z(N)_d / Z(N)_0.$$

With these partition functions and the similar ones on the Gromov-Witten side, Okounkov and Pandharipande [24] prove the local versions of the three MNOP conjectures, which we state. The first concerns the degree 0 Donaldson-Thomas partition function, which as in the absolute case is described in terms of the MacMahon function  $M(q)$ .

**Theorem 1.** *The degree 0 local Donaldson-Thomas partition function is determined by*

$$Z(N; q)_0 = M(-q)^{\int_N c_3(T_N \otimes K_N)}.$$

The integral in the exponent is here defined by (classical)  $T$ -localization on  $N$ ,

$$\int_N c_3(T_N \otimes K_N) = \int_C \frac{c_3(T_N \otimes K_N)}{e(N)}.$$

Notice that the second integral is over the proper curve  $C$  and that the normal bundle of  $C$  in  $N$  is  $N$  itself, so we divide by  $e(N)$  in localization.

Secondly, the reduced local Donaldson-Thomas series satisfy a rationality condition:

**Theorem 2.** *The reduced series  $Z'(N; q, t_1, t_2)_d$  is a rational function in the variables  $t_1, t_2$  and  $q$ .*

Thirdly, the local Gromov-Witten and the local Donaldson-Thomas theories are shown to be equivalent. Let the splitting of  $N$  be

$$N = L_1 \oplus L_2,$$

where  $L_i$  are line bundles on  $C$ , and define  $k_i$  to be the degree of  $L_i$ . Let  $g$  be the genus of  $C$ . With these notations, we have

**Theorem 3.** *After the change of variables  $e^{iu} = -q$ ,*

$$(-iu)^{d(2-2g+k_1+k_2)} Z'_{GW}(N)_d = (-q)^{-\frac{d}{2}(2-2g+k_1+k_2)} Z'_{DT}(N)_d.$$

All of these theorems are formulated and proved in [24], as a byproduct of the complete solution of the local Donaldson-Thomas theory of curves. The strategy is to use certain so-called degeneration formulas for the theory. These formulas express the partition function of invariants over a curve by partition functions over curves of lower genus. Repeated application of these reduces the problem to that of describing the local Donaldson-Thomas theory of  $\mathbb{P}^1$ , which is subsequently solved.

## 2.3 Toric threefolds

One class of examples especially suited to localization techniques are the toric threefolds. Throughout this section, we shall let  $X$  be a smooth, complete

toric threefold, and let  $T$  be the three-dimensional torus acting on  $X$ . In the articles [21] and [22] the foundations are laid for calculating the Donaldson-Thomas invariants of such  $X$  by localization. We will present some parts of the articles, as an example of how toric localization may be used to compute integrals of virtual classes.

### $T$ -fixed ideal sheaves

The first step in employing localization techniques is to describe the  $T$ -fixed part of the space of ideal sheaves  $I_n(X, \beta)^T \subset I_n(X, \beta)$ . We begin by noting that as  $X$  is toric, smooth and proper, there is a convex polyhedron

$$\Delta(X) \subset \mathbb{R}^3$$

associated to  $X$ . The vertices of  $\Delta(X)$  correspond bijectively to  $T$ -fixed points of  $X$ . The fixed point scheme  $X^T \subset X$  is a set of isolated points  $\{X_\alpha\}$ .<sup>1</sup> For each such point  $X_\alpha$ , there is a canonical,  $T$ -invariant, open affine

$$U_\alpha \cong \mathbb{A}^3$$

centered at  $X_\alpha$ . For every such  $U_\alpha$  we may choose coordinates  $t_i$  on  $T$  and  $x_i$  on  $U_\alpha$  such that the  $T$ -action is such that the map  $T \times U_\alpha \rightarrow U_\alpha$  is

$$((t_1, t_2, t_3), (x_1, x_2, x_3)) \mapsto (t_1^{-1}x_1, t_2^{-1}x_2, t_3^{-1}x_3).$$

The edges in the Newton polyhedron  $\Delta(X)$  correspond to the  $T$ -invariant 1-dimensional subschemes of  $X$ . Precisely, if we have a  $T$ -invariant line  $C_{\alpha\beta} \cong \mathbb{P}^1$  incident to two points  $X_\alpha$  and  $X_\beta$ , it corresponds to an edge in  $\Delta(X)$  connecting the two vertices  $X_\alpha$  and  $X_\beta$ . To every such line  $C_{\alpha\beta}$  we assign integers  $m_{\alpha\beta}$  and  $m'_{\alpha\beta}$  by saying that the normal bundle of  $C_{\alpha\beta}$  in  $X$  is

$$N_{C_{\alpha\beta}/X} \cong \mathcal{O}(m_{\alpha\beta}) \oplus \mathcal{O}(m'_{\alpha\beta}).$$

We are now in a position to describe the  $T$ -fixed ideal sheaves. If  $[\mathcal{I}] \in I_n(X, \beta)$  is  $T$ -fixed, the same must be true of the closed subscheme  $Z \subset X$  associated to  $\mathcal{I}$ . Hence, as  $Z$  has dimension at most 1, it must be supported on the  $T$ -fixed points  $X_\alpha$  of  $X$  together with the  $T$ -invariant lines  $C_{\alpha\beta}$  between them.

Over an open affine  $U_\alpha$  with standard coordinates matching the torus action as above, the ideal

$$I_\alpha = \mathcal{I}|_{U_\alpha} \subset \mathbb{C}[x_1, x_2, x_3]$$

---

<sup>1</sup>We follow [21], [22] in using Greek letters for indexing fixed points of  $X$ . As a consequence  $\beta$  is doing double duty as an index and as a homology class, but no confusion should occur from this.

must be  $T$ -fixed. If we have a polynomial  $f \in I_\alpha$  such that

$$f = \sum_{k \in \mathbb{Z}^3} c_k x^k,$$

by the  $T$ -invariance of  $I$ , we see that for every triple  $a = (a_1, a_2, a_3) \in (\mathbb{C}^*)^3$  the polynomial

$$f_a = \sum_{k \in \mathbb{Z}^3} a^{-k} c_k x^k$$

lies in  $I_\alpha$  as well. We see that by taking a suitable  $\mathbb{C}$ -linear combination of such  $f_a$  we may obtain every monomial appearing in  $f$ . As every monomial in  $f$  lies in  $I_\alpha$ , this ideal is generated by monomials.

**Definition 6.** By a  $n$ -dimensional partition we mean a subset  $P$  of  $\mathbb{Z}_{\geq 0}^n$ . We demand that if

$$(a_1, \dots, a_n) \in P$$

and

$$(b_1, \dots, b_n) \in \mathbb{Z}_{\geq 0}^n$$

is such that  $b_i \leq a_i$  for all  $i$ , we have

$$(b_1, \dots, b_n) \in P.$$

We mention that this is in conflict with more classical definitions of partition, where the dimension of a partition is one lower than the dimension we use. For example, MacMahon referred to our 3-dimensional partitions as plane partitions, and what is commonly known as a partition is in our use a 2-dimensional partition. We choose to use this convention in order to highlight the fact that  $n$ -dimensional partitions corresponds to ideals containing what is essentially  $n$ -dimensional information.

The fact that  $I_\alpha$  is generated by monomials allows us to describe it completely by giving the three-dimensional partition

$$\pi_\alpha = \left\{ (k_1, k_2, k_3) \mid x_1^{k_1} x_2^{k_2} x_3^{k_3} \notin I_\alpha \right\} \subset \mathbb{Z}_{\geq 0}^3.$$

Along each coordinate axis of  $\mathbb{Z}^3$ , the partition  $\pi_\alpha$  is described by a two-dimensional partition. Specifically, in the direction corresponding to the  $T$ -invariant curve  $C_{\alpha\beta}$  we have the partition

$$\lambda_{\alpha\beta} = \left\{ (k_2, k_3) \mid \forall k_1, x_1^{k_1} x_2^{k_2} x_3^{k_3} \notin I_\alpha \right\}.$$

This is equivalent to saying that

$$\lambda_{\alpha\beta} = \left\{ (k_2, k_3) \mid x_2^{k_2} x_3^{k_3} \notin I_{\alpha\beta} \right\},$$



where

$$I_{\alpha\beta} = \mathcal{I}|_{U_\alpha \cap U_\beta} \subset \mathbb{C}[x_1^{-1}, x_1, x_2, x_3].$$

Matching up these partitions, we see that the data specifying a  $T$ -invariant ideal sheaf on  $X$  can be organized as

- A three dimensional partition  $\pi_\alpha$  for every fixed point  $X_\alpha$
- A two-dimensional partition  $\lambda_{\alpha\beta}$  for every  $T$ -invariant line  $C_{\alpha\beta}$ , compatible with the partitions  $\pi_\alpha$  in the sense that the asymptotic two-dimensional partition of  $\pi_\alpha$  along the axis corresponding to  $C_{\alpha\beta}$  is  $\lambda_{\alpha\beta}$ .

### Degree and Euler characteristic

The discrete invariants  $n$  and  $\beta$  occurring in the definitions of the Hilbert scheme  $I_n(X, \beta)$  are easy to calculate from the combinatorial data  $\{\pi_\alpha, \lambda_{\alpha\beta}\}$ . We let  $|\lambda_{\alpha\beta}|$  be the size of the partition  $\lambda_{\alpha\beta}$ , defined simply as the number of elements (or “boxes”) in the partition. Then we see that

$$\beta = \sum |\lambda_{\alpha\beta}| [C_{\alpha\beta}].$$

The size of a three-dimensional partition can be similarly defined as the number of boxes. If there are  $T$ -invariant lines incident to  $X_\alpha$ , the partition  $\pi_\alpha$  will be infinite along one of the coordinate axes, making the size so defined infinite. Hence we introduce the renormalized size  $|\pi_\alpha|$ , defined as follows. If the asymptotics of  $\pi_\alpha$  along the coordinate axes are  $\lambda_{\alpha\beta_i}$ , we let

$$|\pi_\alpha| = \#\{\pi_\alpha \cap [0, \dots, N]\} - (N+1) \sum_{i=1}^3 |\lambda_{\alpha\beta_i}|, \quad N \gg 0.$$

The volume defined in this way may be negative.

Given  $m, m' \in \mathbb{Z}$  and a two-dimensional partition  $\lambda$ , we define

$$f_{m,m'}(\lambda) = \sum_{(i,j) \in \lambda} (-mi - m'j + 1).$$

Every edge of  $\Delta(X)$  is assigned a pair of integers  $(m_{\alpha\beta}, m'_{\alpha\beta})$ , determined by the splitting of the normal bundle

$$N_{C_{\alpha\beta}/X} = \mathcal{O}(m_{\alpha\beta}) \oplus \mathcal{O}(m'_{\alpha\beta})$$

We define

$$f(\alpha, \beta) = f_{m_{\alpha\beta}, m'_{\alpha\beta}}(\lambda_{\alpha\beta}).$$

If the ideal sheaf  $\mathcal{I}$  is determined by the partition data  $\{\pi_\alpha, \lambda_{\alpha\beta}\}$  and the subscheme associated to  $\mathcal{I}$  is  $Z$ , we have

$$\chi(\mathcal{O}_Z) = \sum_{\alpha} |\pi_\alpha| + \sum_{\alpha, \beta} f(\alpha, \beta).$$

The proof of this is a simple computation in Čech-cohomology, using the open covering of  $Z$  induced by intersecting the elements of the covering  $\{U_\alpha\}$  with  $Z$ .

### The obstruction theory

We let the Donaldson-Thomas obstruction theory of  $I_n(X, \beta)$  be

$$E_0 \rightarrow E_1,$$

and note that it can be checked that this is  $T$ -equivariant, as per the requirements of [15]. We assume that the virtual dimension of  $I_n(X, \beta)$  is 0, and want to apply the virtual localization formula. The virtual normal bundle on  $I_n(X, \beta)^T$  is

$$E_0^m \rightarrow E_1^m,$$

where  $E_i^m$  is the part of  $E_i$  (restricted to  $I_n(X, \beta)^T$ ) where  $T$  has nontrivial action. Taking the Euler class of this gives us

$$e(N_{I_n(X, \beta)^T})^{\text{vir}} = \frac{e(E_0^m)}{e(E_1^m)}.$$

The induced obstruction theory on  $I_n(X, \beta)^T$  is the  $T$ -fixed part of the original obstruction theory restricted to  $I_n(X, \beta)^T$ . Letting  $S(\mathcal{I})$  be the closed subscheme of  $I_n(X, \beta)^T$  with support in  $[\mathcal{I}]$ , this obstruction theory gives a virtual class  $[S(\mathcal{I})]^{\text{vir}}$ . Applying the virtual localization formula then gives

$$\int_{[I_n(X, \beta)]^{\text{vir}}} 1 = \sum_{[\mathcal{I}] \in I_n(X, \beta)^T} \int_{[S(\mathcal{I})]^{\text{vir}}} \frac{e(E_1^m)}{e(E_0^m)}.$$

In the toric case it is shown in [21] that  $S(\mathcal{I})$  is a closed point, and that the  $T$ -fixed obstruction theory is trivial. As a consequence, the virtual class  $[S(\mathcal{I})]^{\text{vir}}$  is trivial, and the moving part of the virtual normal bundle is the entire bundle. We have the exact sequence of sheaves on  $I_n(X, \beta)$

$$0 \rightarrow T_{I_n(X, \beta)} \rightarrow E_0 \rightarrow E_1 \rightarrow \text{ob} \rightarrow 0.$$

It can be shown that the Zariski tangent space at the point  $[\mathcal{I}]$  is  $\text{Ext}^1(\mathcal{I}, \mathcal{I})$ , and the fibre of the obstruction sheaf over  $[\mathcal{I}]$  is  $\text{Ext}^2(\mathcal{I}, \mathcal{I})$ . Hence over  $[\mathcal{I}]$  we get

$$\frac{e(E_1^m)}{e(E_0^m)} = \frac{e(E_1)}{e(E_0)} = \frac{e(\text{Ext}^2(\mathcal{I}, \mathcal{I}))}{e(\text{Ext}^1(\mathcal{I}, \mathcal{I}))}.$$

This gives us the following formulation of the virtual localization formula:

$$\int_{[I_n(X, \beta)]^{\text{vir}}} 1 = \sum_{[\mathcal{I}] \in I_n(X, \beta)^T} \frac{e(\text{Ext}^2(\mathcal{I}, \mathcal{I}))}{e(\text{Ext}^1(\mathcal{I}, \mathcal{I}))}.$$

As the Euler classes in the above formula are the equivariant ones, the evaluation involves an examination of the action of  $T$  on the spaces  $\text{Ext}^1(\mathcal{I}, \mathcal{I})$  and  $\text{Ext}^2(\mathcal{I}, \mathcal{I})$ . We will not go into the details of such a calculation, instead we mention one case in which the evaluation of the Euler classes is especially simple.

We let  $S$  be a complete toric surface with an effective anticanonical divisor, and consider the total space of the bundle  $K_S$ . This has an embedding into the projective bundle  $X = \mathbb{P}(\mathcal{O}_S \oplus K_S)$ , and it is shown in [21, Sec 3.2] that  $X$  has an anticanonical section. Thus the Donaldson-Thomas theory of  $X$  is well defined, and we may define the reduced Donaldson-Thomas partition function of the surface  $S$  by

$$Z'_{DT}(S; q)_\beta = Z'_{DT}(X; q)_\beta$$

for  $\beta \in H_2(S; \mathbb{Z})$ . Let  $D = X \setminus K_S$ , the divisor at infinity. As  $\beta$  is a class on  $S$ , it can be shown that the support of the curve  $Z$  associated to  $T$ -fixed ideal sheaf must be in  $K_S$ , except possibly for a finite union of 0-dimensional subschemes supported on  $D$ .

We note that if  $\mathcal{I}$  and  $\mathcal{J}$  have associated subschemes with disjoint support, and letting  $\mathcal{K} = \mathcal{I} \oplus \mathcal{J}$ , we get

$$\text{Ext}^i(\mathcal{K}, \mathcal{K}) = \text{Ext}^i(\mathcal{I}, \mathcal{I}) \oplus \text{Ext}^i(\mathcal{J}, \mathcal{J}).$$

This implies the following relation on the Euler classes:

$$\frac{e(\text{Ext}^2(\mathcal{K}, \mathcal{K}))}{e(\text{Ext}^1(\mathcal{K}, \mathcal{K}))} = \frac{e(\text{Ext}^2(\mathcal{I}, \mathcal{I}))}{e(\text{Ext}^1(\mathcal{I}, \mathcal{I}))} \cdot \frac{e(\text{Ext}^2(\mathcal{J}, \mathcal{J}))}{e(\text{Ext}^1(\mathcal{J}, \mathcal{J}))}$$

Putting together the above facts shows that

$$\begin{aligned} Z_{DT}(X; q)_\beta &= \sum_n q^n \sum_{[\mathcal{I}] \in I_n(X, \beta)^T} \frac{e(\text{Ext}^2(\mathcal{I}, \mathcal{I}))}{e(\text{Ext}^1(\mathcal{I}, \mathcal{I}))} \\ &= \left( \sum_n q^n \sum_{[\mathcal{I}] \in I_n(K_S, \beta)^T} \frac{e(\text{Ext}^2(\mathcal{I}, \mathcal{I}))}{e(\text{Ext}^1(\mathcal{I}, \mathcal{I}))} \right) \left( \sum_n q^n \sum_{[\mathcal{I}] \in I_n(D, 0)^T} \frac{e(\text{Ext}^2(\mathcal{I}, \mathcal{I}))}{e(\text{Ext}^1(\mathcal{I}, \mathcal{I}))} \right). \end{aligned}$$

The same equation holds for the degree 0 series  $Z_{DT}(X; q)_0$ , replacing  $\beta$  with 0 everywhere. This gives us

$$\begin{aligned} Z'_{DT}(S; q)_\beta &= Z_{DT}(X; q)_\beta / Z_{DT}(X; q)_0 \\ &= \frac{\sum_n q^n \sum_{[\mathcal{I}] \in I_n(K_S, \beta)^T} e(\text{Ext}^2(\mathcal{I}, \mathcal{I})) / e(\text{Ext}^1(\mathcal{I}, \mathcal{I}))}{\sum_n q^n \sum_{[\mathcal{I}] \in I_n(K_S, 0)^T} e(\text{Ext}^2(\mathcal{I}, \mathcal{I})) / e(\text{Ext}^1(\mathcal{I}, \mathcal{I}))}. \end{aligned}$$

In particular we see that the Donaldson-Thomas theory of  $S$  does not depend on the compactification chosen.

In this case of a local Calabi-Yau threefold we have the following simple evaluation of the Euler class of the virtual normal bundle in  $[\mathcal{I}]$ . ([21, theorem 2].)

**Theorem 4.** *Let  $\mathcal{I}$  be a  $T$ -fixed ideal sheaf in  $I_n(K_S, \beta)$ , such that the closed subscheme associated to  $\mathcal{I}$  is  $Z$ . Then*

$$\frac{e(\mathrm{Ext}^2(\mathcal{I}, \mathcal{I}))}{e(\mathrm{Ext}^1(\mathcal{I}, \mathcal{I}))} = (-1)^{\chi(\mathcal{O}_Z) + \sum_{\alpha\beta} m_{\alpha\beta} |\lambda_{\alpha\beta}|}$$

where the sum in the exponent is over all edges and

$$\mathcal{O}(m_{\alpha\beta}) \oplus \mathcal{O}(m'_{\alpha\beta})$$

is the normal bundle to the edge curve  $C_{\alpha\beta}$ .

Note in particular the case where  $\mathcal{I}$  is the ideal sheaf of a zero-dimensional subscheme of length  $n$ , where we get

$$\frac{e(\mathrm{Ext}^2(\mathcal{I}, \mathcal{I}))}{e(\mathrm{Ext}^1(\mathcal{I}, \mathcal{I}))} = (-1)^n.$$

Recall the MacMahon function  $M(q)$  which is the generating function for the number of three-dimensional partitions with  $k$  elements. On a toric threefold  $X$ , the number of fix-points is equal to the Euler characteristic  $\chi(X)$ . As  $T$ -invariant 0-dimensional subschemes are described by a three-dimensional partition in each fixed point, we see that  $M(q)^{\chi(X)}$  is the generating function for the number of  $T$ -fixed subschemes of  $X$  of length  $k$ . This can then together with the equation above be used to give a proof of the first MNOP conjecture in the case of toric Calabi-Yau threefolds, i.e.

$$Z_{DT}(X; q)_0 = \sum_{n \in \mathbb{Z}} D_{n,0} = M(-q)^{\chi(X)}.$$

## 2.4 Weighted Euler characteristics

In the article [3] Behrend introduced a new tool for calculating integrals of virtual fundamental classes. Before stating the results, we need a few definitions.

**Definition 7.** *A constructible subset of a scheme  $Y$  is one that is obtained from subschemes and finitely many uses of the set-theoretic operations of union and intersection. A constructible function on a scheme  $Y$  is one such that  $Y$  has a finite partition into constructible subsets such that the function is constant on each constructible subset.*

We let  $Y$  be any scheme, where the typical example to bear in mind is the one where  $Y$  is a moduli space with a perfect obstruction theory on it. This obstruction theory should further be *symmetric*. This is a property which for example is fulfilled by the obstruction theory defined on the Hilbert scheme  $I_n(X, \beta)$  when  $X$  is a Calabi-Yau threefold.

On such a scheme  $Y$ , Behrend constructs a canonical constructible function  $\nu_Y : Y \rightarrow \mathbb{Z}$ . Using this function we may define a new invariant of  $Y$  as follows.

**Definition 8.** Given a scheme  $Y$ , we define the Euler characteristic of  $Y$  relative to a constructible function  $f : Y \rightarrow \mathbb{Z}$  to be

$$\chi(Y, f) = \sum_{n \in \mathbb{Z}} n \chi(f^{-1}(n)).$$

We define the weighted Euler characteristic of  $Y$  to be

$$\tilde{\chi}(Y) = \chi(Y, \nu_Y) = \sum_{n \in \mathbb{Z}} n \chi(\nu_Y^{-1}(n)).$$

We mention a few properties of the function  $\nu_Y$ .

- If  $Y$  is smooth in  $P$ , we have  $\nu_Y(P) = (-1)^{\dim Y}$ .
- Multiplicativity holds:  $\nu_{Y \times Z}(P, Q) = \nu_Y(P) \nu_Z(Q)$ .
- The function  $\nu_Y$  is an invariant of the analytic structure of  $Y$ . See [3, Lemma 4.22]. As a consequence, if  $f : Y \rightarrow Z$  is an étale morphism, we have  $\nu_Y = f^* \nu_Z$ .

### Symmetric obstruction theories

We assume now that  $Y$  has a perfect obstruction theory defined on it. Furthermore, we demand that this obstruction theory be symmetric.

**Definition 9.** A perfect obstruction theory  $E \rightarrow L_X$  is called symmetric, if there is an isomorphism  $\theta : E \rightarrow E^*[1]$ , satisfying  $\theta^*[1] = \theta$ .

The morphism  $\theta$  is here a morphism in the derived category, and the dual complex  $E^*$  is the dual of  $E$  in the derived category sense. The  $[1]$  denotes shifting the complex one place to the left.

As our complex  $[E^{-1} \rightarrow E^0]$  has components that are locally free, the dual complex is obtained by taking the component-wise dual. Hence

$$E^*[1] = (E^0)^* \rightarrow (E^{-1})^*$$

lying in degrees  $-1$  and  $0$ . We note a few consequences of an obstruction theory being symmetric. First of all, by the isomorphism  $\theta$  we have the following equations on the ranks of  $E$  and  $E^*[1]$ :

$$\mathrm{rk} E^0 - \mathrm{rk} E^{-1} = \mathrm{rk} (E^{-1})^* - \mathrm{rk} (E^0)^* = -(\mathrm{rk} E^0 - \mathrm{rk} E^{-1}).$$

Hence

$$\mathrm{rk} E = \mathrm{rk} E^0 - \mathrm{rk} E^{-1} = 0,$$

so the virtual dimension of  $Y$  is 0.

Another consequence of an obstruction theory being symmetric is an isomorphism between the obstruction sheaf  $\text{ob}$  and the cotangent sheaf  $\Omega_Y$ . Namely, we have

$$\text{ob} = h^1(E^*) = h^0(E^*[1]) = h^0(E) = \Omega_Y,$$

where the last equality follows from one of the conditions for  $E$  being an obstruction theory.

As the virtual dimension of  $Y$  is 0 whenever we have a perfect symmetric obstruction theory on  $Y$ , we get a virtual fundamental class  $[Y]^{\text{vir}}$  of dimension 0 on  $Y$ . Hence we have the virtual count of  $Y$ :

$$\#^{\text{vir}}(Y) = \int_{[Y]^{\text{vir}}} 1.$$

The main result of the paper [3] is that this virtual count is equal to the weighted Euler characteristic, that is

$$\#^{\text{vir}}(Y) = \tilde{\chi}(Y).$$

One interesting consequence of this is that the virtual count on  $Y$  depends only on the scheme structure of  $Y$  and not on the particular perfect symmetric obstruction theory used to define it.

### Donaldson-Thomas invariants of super-rigid curves

As we have seen, the Donaldson-Thomas invariants  $D_{n,\beta}$  of a threefold  $X$  with trivial canonical class is expressible as the weighted Euler characteristic  $\tilde{\chi}(I_n(X, \beta))$ . An application of this is given by Behrend and Bryan in [4]. Given a Calabi-Yau threefold  $X$ , they calculate the contribution from super-rigid rational curves on  $X$  to the Donaldson-Thomas invariants of  $X$ .

We say a smoothly embedded rational curve  $C$  on a Calabi-Yau threefold  $X$  is super-rigid if the normal bundle of  $C$  in  $X$  is

$$N_{C/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1).$$

Let  $E$  be the effective cycle  $\sum n_i C_i$  where  $C_i$  are pairwise disjoint super-rigid curves, and  $n_i > 0$ . We let  $\beta$  be the class of  $E$  in homology. We may consider the one-dimensional subschemes  $Z \subset X$  such that the associated cycle of  $Z$  is  $E$ . It can be shown that such one-dimensional subschemes form an open and closed subscheme

$$J_n(X, E) \subset I_n(X, \beta).$$

This is essentially because superrigidity shows that the  $C_i$  have no infinitesimal deformations, and hence the fundamental cycle of a subscheme  $Z$  with associated cycle  $E$  must be fixed when  $Z$  varies continuously in  $I_n(X, \beta)$ . See [4, Remark 2.2].

As  $J_n(X, E)$  is open in  $I_n(X, \beta)$ , the symmetric obstruction theory on  $I_n(X, \beta)$  induces by restriction a symmetric obstruction theory on  $J_n(X, E)$ , giving a virtual fundamental class  $[J_n(X, E)]^{\text{vir}}$  of dimension 0. Since  $J_n(X, E)$  is closed in  $I_n(X, \beta)$ , it is proper and we may integrate the virtual fundamental class over  $J_n(X, E)$ , giving an integer

$$D_{n,E} = \#^{\text{vir}}(J_n(X, E)) = \int_{[J_n(X, E)]^{\text{vir}}} 1.$$

We call this number the contribution of  $J_n(X, E)$  to  $D_{n,\beta}$ .

A subscheme  $Z \subset X$  lying in  $J_n(X, E)$  consists of the union of subschemes of two types:

1. Curves supported on  $C_i$ , possibly with embedded points
2. 0-dimensional subschemes with support disjoint from each  $C_i$ .

The contribution from  $J_n(X, E)$  to the Donaldson-Thomas invariant is calculated in the following manner. We may stratify  $J_n(X, E)$  using products of spaces each containing only subschemes of type 1 or 2. Hence we may express the weighted Euler characteristic  $\tilde{\chi}(J_n(X, E))$  by Euler characteristics relative to  $\nu_{J_n(X, E)}$  of spaces containing only subschemes of type 1 or 2.

Let  $\tilde{J}_m(X, C_i)$  be the closed subset of  $J_m(X, E)$  consisting of subschemes supported on  $C_i$ . (We use  $m$  rather than  $n$  here because we are considering only a component of a scheme  $[Z] \in J_n(X, E)$ ). First note that there is an analytic neighborhood of  $C_i$  isomorphic to the total space of the bundle on  $\mathbb{P}^1$

$$N = N_{C_i/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1).$$

Because of this, and because the weight function  $\nu_{J_m(X, E)}$  only depends on the analytic structure on  $J_m(X, E)$ , it can be shown that we may calculate the weighted Euler characteristic  $\chi(\tilde{J}_m(X, C_i), \nu_{J_m(X, E)})$  as if  $C_i$  were the zero section of  $N$ . As  $N$  is toric, this can then be handled by localization, giving an expression for  $\chi(J_m(X, C_i), \nu_{J_m(X, E)})$ .

For the spaces of subschemes of type 2, which are Hilbert schemes of points on  $X$ , the weighted Euler characteristic is already computed in a different article [6]. Stratifying  $J_n(X, E)$  by products of spaces of these two types, it is possible to make a complete calculation of the contribution from  $J_n(X, E)$  to  $D_{n,\beta}$ .

A parallel definition of the contribution from curves with cycle  $E$  can be given on the Gromov-Witten side. There is an open and closed component  $\overline{M}_g(X, E) \subset \overline{M}_g(X, \beta)$  of curve maps sending the fundamental cycle of the curve to the cycle  $E$ . We can then in the same manner define the contribution  $N_{g,E}$  of  $\overline{M}_g(X, E)$  to the Gromov-Witten invariant to be the integral of the virtual fundamental class  $[\overline{M}_g(X, E)]^{\text{vir}}$ . There is a Donaldson-Thomas/Gromov-Witten correspondence for contributions from super-rigid

curves which parallels the usual correspondence (Conjecture 3), replacing  $D_{n,\beta}$  and  $N_{g,\beta}$  with  $D_{n,E}$  and  $N_{g,E}$  in the definitions.

Using the calculation of Donaldson-Thomas invariants from super-rigid curves above, this DT/GW-correspondence is shown to hold. As a consequence of the super-rigid correspondence we get the following corollary.

**Corollary 1.** *Let  $X$  be a Calabi-Yau threefold, and let  $\beta \in H_2(X; \mathbb{Z})$  be a curve class such that all cycle representatives of  $\beta$  are supported on a collection of pairwise disjoint, super-rigid rational curves. Then the degree  $\beta$  DT/GW-correspondence holds, that is*

$$Z'_{GW}(X; u)_{\beta} = Z'_{DT}(X; -e^{iu})_{\beta}.$$

As a concrete example, we get

**Corollary 2.** *Let  $X \subset \mathbb{P}^4$  be a quintic threefold, and let  $L$  be the class of a line. Then, for  $\beta$  equal to  $L$  or  $2L$ , the DT/GW-correspondence holds.*

*Proof.* By the deformation invariance of both DT- and GW-invariants, we may assume that  $X$  is a generic quintic threefold. It is known that there are 2875 pairwise disjoint lines on  $X$ , and that these are super-rigid. Similarly it is known that there are 609250 pairwise disjoint conics on  $X$ , and that these are super-rigid as well. As curves of class  $L$  or  $2L$  are either lines, conics or unions of lines, the conditions of Corollary 1 hold.  $\square$

It is known for all degrees  $d < 5$  that there are finitely many rational curves of degree  $d$  on a generic quintic  $X$ , each smoothly embedded and super-rigid. However, one still cannot prove the Donaldson-Thomas/Gromov-Witten correspondence for  $\beta = kL$  with  $k > 3$  by Corollary 1. The reason for this is simply that in degree 3 and above we get contributions from elliptic curves on  $X$ , causing the conditions of Corollary 1 to fail.



## Chapter 3

# Elliptic fibrations

### 3.1 Trivial elliptic fibrations

In the article [10] Edidin and Qin investigate the GW/DT-correspondence for the special case where the threefold is a trivial elliptic fibration. By a trivial elliptic fibration we mean a threefold  $X$  that is isomorphic to  $E \times S$ , where  $E$  is an elliptic curve and  $S$  is a smooth surface.

In a previous article by Katz, Li and Qin Donaldson-Thomas invariants were computed for the moduli spaces  $I_0(X, d\beta_0)$  and  $I_1(X, d\beta_0)$ , where  $\beta_0$  is the class of a fibre of  $E \times S \rightarrow S$ . The moduli spaces involved were shown to be smooth, allowing the virtual counts to be expressed up to sign as Euler characteristics, which were then computed.

The question answered by the main theorem of [10] concerns the general GW/DT-correspondence with primary insertions, see Definitions 3 and 5. We state this conjecture here as it appears in the article.

**Conjecture 4.** *Let  $\beta \in H_2(X; \mathbb{Z}) \setminus \{0\}$ , and  $d = -\int_{\beta} K_X$ . Then after the change of variables  $e^{iu} = -q$ , we have*

$$(-iu)^d Z'_{GW} \left( X; u \mid \prod_{i=1}^r \tau_0(\gamma_i) \right)_{\beta} = (-q)^{-d/2} Z'_{DT} \left( X; q \mid \prod_{i=1}^r \tilde{\tau}_0(\gamma_i) \right)_{\beta}.$$

Under some conditions on  $S$  or on the intersection classes  $\gamma_i$ , this conjecture is shown to hold. Precisely, we have the following theorem.

**Theorem 5.** *Let  $f : X = E \times S \rightarrow S$  be the projection where  $E$  is an elliptic curve and  $S$  is a smooth surface and  $\beta \in H_2(X; \mathbb{Z})$ . Then the Gromov-Witten/Donaldson-Thomas correspondence above holds if either  $\int_{\beta} K_X = \int_{\beta} f^* K_S = 0$ , or*

$$\gamma_1, \dots, \gamma_r \in f^* H^*(S; \mathbb{Q}) \subset H^*(X; \mathbb{Q}).$$

As a special case we get the following corollary.

**Corollary 3.** *Let  $E$  be an elliptic curve and  $S$  a smooth surface with numerically trivial canonical class  $K_S$ . Then the Gromov-Witten/Donaldson-Thomas correspondence above holds for the threefold  $X = E \times S$ .*

We give a summary of the arguments of the article, which provide an example of equivariant theory being used in a different setting than toric localization. For simplicity we restrict to the case of  $r = 0$ , that is of the DT- and GW-invariants without insertions  $D_{n,\beta}$  and  $N_{g,\beta}$ . Throughout this chapter, let  $X$  be a trivial elliptic fibration, and let  $f : X = E \times S \rightarrow S$  be the projection map. We let  $\beta_0 \in H_2(X, \mathbb{Z})$  be the class of a fibre of  $f$ . We fix an identity point  $e \in E$ , giving  $E$  the structure of a group, which has a natural action on  $X$ .

The two sides of the Donaldson-Thomas/Gromov-Witten correspondence are each calculated separately, and afterwards shown to match as conjectured. The overall strategies of the two cases are similar, we shall focus here on the Donaldson-Thomas case.

The main idea of the paper is to make use of the group action of  $E$  on  $X$ . This action naturally induces an  $E$ -action on the Hilbert scheme  $I_n(X, \beta)$ . We first note the following.

**Lemma 1.** *If  $n \neq 0$  or  $\beta \neq d\beta_0$  for some integer  $d$ , the action of  $E$  on  $I_n(X, \beta)$  has no fixed points.*

*Proof.* If  $[Z]$  is an  $E$ -fixed point in  $I_n(X, \beta)$ , we see that the associated closed subscheme  $Z \subset X$  is also  $E$ -fixed. For this to be true  $Z$  must be the inverse image under  $f$  of some subscheme  $Q$  of  $S$ . Since  $Z$  is at most one-dimensional,  $Q$  must be zero-dimensional. Taking  $d$  to be the length of  $Q$ , we find  $n = \chi(\mathcal{O}_Z) = 0$ , and  $\beta = d\beta_0$ .  $\square$

**Lemma 2.** *Let  $n$  and  $\beta$  be such that  $E$  acts without fixed points on  $I_n(X, \beta)$ . Then*

$$D_{n,\beta} = 0.$$

*Proof.* The stabilizer of a point  $[Z] \in I_n(X, \beta)$  under the action of  $E$  is a closed subscheme of  $E$ . As there are no fixed points, this stabilizer cannot be  $E$  itself, and so must be a finite set of points in  $E$ . We wish to show that there is an  $N > 0$  such that for every  $[Z] \in I_n(X, \beta)$ , the cardinality of the stabilizer of  $[Z]$  is less than  $N$ . We note that the stabilizers of points in  $I_n(X, \beta)$  together form a closed subscheme

$$Y \subset E \times I_n(X, \beta)$$

such that the fibre  $Y_{[Z]}$  over a point  $[Z] \in I_n(X, \beta)$  is, considered as a subscheme of  $E$ , the stabilizer of  $[Z]$ . As  $I_n(X, \beta)$  is proper the image of  $Y$

under the projection to  $E$ , which is the same as the union of all stabilizer subgroups of  $E$ , is a closed subset of  $E$ . Since this image is the union of finite subgroups of  $E$ , it must be strictly smaller than  $E$ . Hence the union of all stabilizer subgroups is a finite set of points, and we may take  $N$  to be this number of points.

It is clear that for any cyclic subgroup  $G$  of  $E$  such that the order of  $G$  is a prime greater than  $N$ , the action of  $G$  on  $I_n(X, \beta)$  is free.

Since the obstruction theory is equivariant for the action of any algebraic group of automorphisms of  $X$ , the cycle  $[I_n(X, \beta)]^{\text{vir}}$  defines an element of the equivariant Borel-Moore homology group  $H_*^G(I_n(X, \beta); \mathbb{Z})$  defined in [9]. If the virtual dimension of  $I_n(X, \beta)$  is not 0, the invariants  $D_{n, \beta}$  vanish trivially. Hence we may assume that  $[I_n(X, \beta)]^{\text{vir}}$  gives an element in  $H_0^G(I_n(X, \beta); \mathbb{Z})$ . As  $G$  acts freely on  $I_n(X, \beta)$ , it can be shown that any element in this group is represented by a  $G$ -equivariant 0-cycle on  $I_n(X, \beta)$ , whose degree is a multiple of  $p$ . Hence  $p$  divides

$$\deg[I_n(X, \beta)]^{\text{vir}} = D_{n, \beta}$$

and as we may choose  $p$  arbitrarily large, we get  $D_{n, \beta} = 0$ .  $\square$

We note that both of these lemmas work in the Gromov-Witten setting, with the suitable adjustment of discrete invariants in Lemma 1. Specifically, we see that unless  $g = 1$  and  $\beta = d\beta_0$  for  $d \geq 0$ ,  $E$  acts without fixed points on  $\overline{M}_{g,0}(X, \beta)$ . Like in Lemma 2 it can be shown that when  $E$  acts without fixed points on  $\overline{M}_{g,0}(X, \beta)$  the corresponding invariants  $N_{g, \beta}$  vanish.

Hence it is shown, for example, that when  $\beta \neq d\beta_0$ , we have

$$Z'_{DT}(X; q)_\beta = Z'_{GW}(X; u)_\beta = 0,$$

giving the Donaldson-Thomas/Gromov-Witten correspondence in this case.

What remains for a proof of Theorem 5 (restricting to  $r = 0$ ) is to calculate the Donaldson-Thomas invariants  $D_{0, d\beta_0}$  and the Gromov-Witten invariants  $N_{1, d\beta_0}$ . For the Donaldson-Thomas case, we first note that it can be shown that for every scheme  $Z$  on  $X$  lying in  $I_0(X, d\beta_0)$  the ideal sheaf  $\mathcal{I}$  of  $Z$  satisfies

$$\mathcal{I} = f^* \mathcal{J},$$

for some ideal sheaf  $\mathcal{J}$  of a dimension 0 subscheme on  $S$  of length  $d$ . This correspondence gives an isomorphism

$$I_0(X, d\beta_0) \cong S^{[d]}.$$

The virtual count of points in  $I_0(X, d\beta_0)$  is computed by explicitly describing the obstruction sheaf on  $I_0(X, d\beta_0)$ .

**Lemma 3.** *The obstruction bundle over the moduli scheme  $I_0(X, d\beta_0) \cong S^{[d]}$  is isomorphic to the tangent bundle  $T_{S^{[d]}}$  of the Hilbert scheme  $S^{[d]}$ .*

We will not reproduce the proof of this lemma here. We simply note the consequence, namely that as the moduli space is smooth, the Donaldson-Thomas invariant is

$$D_{0,d\beta_0} = (-1)^{\dim S^{[d]}} \deg e(T_{S^{[d]}}) = \chi(S^{[d]}).$$

As the invariants  $D_{n,0}$  vanish for  $n > 0$  the reduced partition function is the same as the unreduced. Putting all these results together, as well as a calculation of the remaining Gromov-Witten invariants, we obtain a full description of the invariants of  $X$ :

$$Z'_{GW}(X; u, v) = Z'_{DT}(X; q, v) = \sum_{d \geq 0} \chi(S^{[d]}) v^{d\beta_0}.$$

### 3.2 Locally trivial elliptic fibrations

Using Behrend's description of the Donaldson-Thomas invariants of  $X$  as a weighted Euler characteristic, we now extend some of the results of the previous chapter to a slightly more general threefold than a trivial elliptic fibration. Specifically, we are interested in the threefolds satisfying the following definition:

**Definition 10.** *Let  $S$  be a proper, smooth surface, and let  $E$  be an elliptic curve. We say a morphism  $f$  from a threefold  $X$  to  $S$  is a locally trivial<sup>1</sup> elliptic fibration over  $S$  with fibre  $E$  if the following criterium is met: There is a covering of  $S$  by analytic open subsets such that for every  $U$  in the covering we have the commutative diagram of analytic spaces*

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{\cong} & E \times U \\ \downarrow f & & \downarrow \text{proj} \\ U & \xlongequal{\quad} & U \end{array} \quad (3.1)$$

Throughout this section we will deal with the Donaldson-Thomas invariants arising from spaces  $I_n(X, d\beta_0)$  where  $X$  is a locally trivial elliptic fibration and  $\beta_0 \in H_2(X; \mathbb{Z})$  is the class of a fibre of the projection  $X \rightarrow S$ . In order to simplify notation slightly, we let  $I_n(X, d) := I_n(X, d\beta_0)$ .

For any analytically open subset  $W \subset X$ , we let  $I_n(W, d) \subset I_n(X, d)$  be the analytically open subset consisting of subschemes with support contained in  $W$ . That this is well defined as an analytic space can be seen by the theory of Douady spaces. These are analogues of the Hilbert scheme in the category of complex analytic spaces, such that for a complex analytic space  $Y$  we have

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<sup>1</sup>The naming of such morphisms is our own, intended to be relevant only for this section, and could possibly be in conflict with some generally accepted definition of what it means for an elliptic fibration to be locally trivial.

a Douady space  $D(Y)$ , which is a complex analytic space parametrizing proper closed analytic subspaces of  $Y$ . When  $Y$  is a projective complex scheme, analytic subspaces of  $Y$  correspond to subschemes by GAGA, and this gives an isomorphism of analytic spaces (see [16, Chapter VIII])

$$\mathrm{Hilb}_{an}(X) \cong D(X_{an})$$

In particular, we may consider  $I_n(W, d)$  as the part of the Douady space  $D(W)$  contained in  $I_n(X, d)_{an}$ .

**Theorem 6.** *Let  $X$  be a proper, smooth threefold admitting a locally trivial elliptic fibration  $f : X \rightarrow S$  with fibre  $E$ , and assume that  $X$  has trivial canonical class. Then*

$$\#^{\mathrm{vir}}(I_0(X, d)) = \chi(S^{[d]}),$$

and for  $n > 0$  we have

$$\#^{\mathrm{vir}}(I_n(X, d)) = 0.$$

We let  $Y$  be the trivial elliptic fibration  $E \times S$  and let  $g : Y \rightarrow S$  be the projection. We note that the theorem holds for the threefold  $Y$  by the results of the previous chapter. As both  $X$  and  $Y$  have trivial canonical divisors, the obstruction theories on  $I_n(X, d)$  and  $I_n(Y, d)$  are symmetric. Hence the virtual count is equal to the weighted Euler characteristic, and to prove the theorem it is enough to show that

$$\chi(I_n(X, d), \nu_{I_n(X, d)}) = \chi(I_n(Y, d), \nu_{I_n(Y, d)}). \quad (3.2)$$

Before proving the above equation we need some lemmas.

**Lemma 4.** *Given two open subsets  $A$  and  $B$  of a topological space  $X$  we have the following formula:*

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).$$

*Proof.* See [27, p. 205]. □

**Lemma 5.** *Let  $A$  and  $B$  be analytically open subsets of a scheme  $Y$ . Then we have*

$$\chi(A \cup B, \nu_Y|_{A \cup B}) = \chi(A, \nu_Y|_A) + \chi(B, \nu_Y|_B) - \chi(A \cap B, \nu_Y|_{A \cap B}).$$

*Proof.* For every  $k \in \mathbb{Z}$ , Lemma 4 gives

$$\begin{aligned} \chi(\nu_Y|_{A \cup B}^{-1}(k)) &= \chi(\nu_Y^{-1}(k) \cap (A \cup B)) \\ &= \chi(\nu_Y^{-1}(k) \cap A) + \chi(\nu_Y^{-1}(k) \cap B) - \chi(\nu_Y^{-1}(k) \cap (A \cap B)) \\ &= \chi(\nu_Y|_A^{-1}(k)) + \chi(\nu_Y|_B^{-1}(k)) - \chi(\nu_Y|_{A \cap B}^{-1}(k)). \end{aligned}$$

Summing the equality between the first and last expression over all  $k$  gives the desired result, using Definition 8.  $\square$

**Lemma 6.** *There exists analytically open coverings  $\{U_i\}_1^k$  and  $\{V_i\}_1^k$  of  $I_n(X, d)$  and  $I_n(Y, d)$  such that for any nonempty set  $J \subset \{1, \dots, k\}$  we have isomorphisms as analytic spaces*

$$\bigcap_{j \in J} U_j \cong \bigcap_{j \in J} V_j.$$

*Proof.* Let  $Z$  be a closed subscheme of  $X$  lying in  $I_n(X, d)$ . The closed subscheme  $f(Z) \subset S$  has support in a finite set of points  $\{p_i\}$ . Choose pairwise disjoint analytically open neighborhoods  $W_i$  for each of these points. As  $X$  has a locally trivial fibration, we may choose the  $W_i$  small enough to satisfy the trivializing diagram 3.1, in particular we have

$$f^{-1}(W_i) \cong E \times W_i.$$

Letting  $W$  be the union of the  $W_i$ , as they are pairwise disjoint, we get

$$f^{-1}(W) \cong E \times W.$$

We also have

$$g^{-1}(W) \cong E \times W,$$

hence there is an isomorphism  $f^{-1}(W) \cong g^{-1}(W)$ .

This isomorphism induces an isomorphism  $I_n(f^{-1}(W), d) \cong I_n(g^{-1}(W), d)$ . This can be seen by considering the two spaces as Douady spaces, so that clearly their complex analytic structure depends only on  $f^{-1}(W)$  and  $g^{-1}(W)$ . As  $I_n(f^{-1}(W), d)$  is a neighborhood of  $[Z]$  it is clear that we may find a collection  $\{W_i\}$  of open subsets of  $S$  such that

$$I_n(X, d) = \bigcup I_n(f^{-1}(W_i), d)$$

and

$$I_n(Y, d) = \bigcup I_n(g^{-1}(W_i), d).$$

As both  $I_n(X, d)$  and  $I_n(Y, d)$  are proper, finitely many  $W_i$  will suffice. Let  $k$  be the number of  $W_i$ 's in this finite collection.

We see that for a nonempty set  $J \subset \{1, \dots, k\}$

$$\bigcap_{j \in J} f^{-1}(W_j) \cong E \times \left( \bigcap_{j \in J} W_j \right) \cong \bigcap_{j \in J} g^{-1}(W_j).$$

We let  $U_i := I_n(f^{-1}(W_i), d)$  and  $V_i = I_n(g^{-1}(W_i), d)$ . Then

$$\begin{aligned} \bigcap_{j \in J} U_j &= \bigcap_{j \in J} I_n(f^{-1}(W_j), d) = I_n(f^{-1}(\bigcap_{j \in J} W_j), d) \\ &\cong I_n(g^{-1}(\bigcap_{j \in J} W_j), d) = \bigcap_{j \in J} I_n(g^{-1}(W_j), d) = \bigcap_{j \in J} V_j, \end{aligned}$$

which is what we wanted to prove.  $\square$

We are now ready to give the proof of (3.2), which gives Theorem 6.

*Proof of (3.2).* Let  $\{U_i\}_1^k$  and  $\{V_i\}_1^k$  be open coverings of  $I_n(X, d)$  and  $I_n(Y, d)$  as in Lemma 6. By [3, Proposition 4.22] the value of  $\nu$  in a point  $P$  depends only on an analytical neighborhood of  $P$ . Hence for  $J \subset \{1, \dots, k\}$  the isomorphisms

$$\bigcap_{j \in J} U_j \cong \bigcap_{j \in J} V_j$$

give equalities of weighted Euler characteristics

$$\chi \left( \bigcap_{j \in J} U_j, \nu_{I_n(X, d)} \right) = \chi \left( \bigcap_{j \in J} V_j, \nu_{I_n(Y, d)} \right).$$

By repeated applications of Lemma 5 we get the following equalities

$$\begin{aligned} \chi(I_n(X, d), \nu_{I_n(X, d)}) &= \sum_{J \subset \{1, \dots, k\}} (-1)^{|J|+1} \chi \left( \bigcap_{j \in J} U_j, \nu_{I_n(X, d)} \right) \\ &= \sum_{J \subset \{1, \dots, k\}} (-1)^{|J|+1} \chi \left( \bigcap_{j \in J} V_j, \nu_{I_n(Y, d)} \right) \\ &= \chi(I_n(Y, d), \nu_{I_n(Y, d)}), \end{aligned}$$

which is what we wanted.  $\square$

It is not obvious that there are any locally trivial elliptic fibrations  $X$  with trivial canonical divisor other than the trivial example of  $E \times S$  with  $K_S = 0$ . One candidate for such an  $X$  would be an abelian threefold. We then require that this threefold has at least one elliptic curve on it, and furthermore that it is not isomorphic to any product  $E \times S$ . We have not been able to prove the existence of such a threefold, though it seems likely that one exists.





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